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(FOUNDED BY ORMOND STONE)

EDITED BY

ORMOND STONE — J. W. ALEXANDER
L. P. EISENHART — T. H. GRONWALL
OSWALD VEBLEN — J. H. M. WEDDERBURN

WITH THE COÖPERATION OF

A. A. BENNETT — H. BLUMBERG
G. A. PFEIFFER — J. K. WHITTEMORE

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THE HISTORY OF NOTATIONS OF THE CALCULUS.

By Florian Cajori.

The Mathematician:
"The symbols he hath found shall sound his praise,
And lead him on through unimagined ways
To conquests new, in worlds not yet created."
James Clerk Maxwell.

1. Introduction.

Without a well-developed notation the differential and integral calculus could not perform its great function in modern mathematics. The history of the growth of the calculus notations is not only interesting, but it may serve as a guide in the invention of fresh notations in the future. The study of the probable causes of the success or failure of past notations may enable us to predict with greater certainty the fate of new symbols which may seem to be required, as the subject gains further development.

Toward the latter part of the eighteenth century the notations of the differential and integral calculus then in vogue no longer satisfied all the needs of this advancing science, either from the standpoint of the new fundamental conceptions that came to be advanced or the more involved analytic developments that followed. Some new symbols were proposed soon after 1750, but the half century of great unrest and much experimentation in matters of calculus notations was initiated by the publication of J. L. Lagrange's Théorie des fonctions analytiques, in 1797.

The fourth volume of Cantor's Vorlesungen über Geschichte der Mathematik, dealing with the period 1759–1799, pays comparatively little attention to these notations. Nor has any one yet published a detailed history of calculus notations for the nineteenth century. The present paper aims to supply a want in this field. Except for certain details, the writer will touch lightly upon the early history of these notations, for that is given quite fully in our histories of mathematics and in the biographies of Newton and Leibniz.*

2. Symbols for Fluxions, Differentials and Derivatives.

a. *Total Differentiation during the Eighteenth Century.*

1. Newton. Newton's earliest use of dots, "pricked letters," to indicate velocities or fluxions is found on a leaf, dated May 20, 1665; no facsimile

* See also F. Cajori, "Spread of calculus notations," in the Bulletin of the Am. Math. Soc., vol. 27, 1921, p. 453-458.

reproduction of it has ever been made.* The earliest printed account of Newton's fluxional notation appeared from his pen in the Latin edition of Wallis' Algebra;† where one finds not only the fluxionary symbols $\dot{x}, \ddot{x}, \ldots$ but also the fluxions of a fraction and radical, thus $\frac{y}{b-x}\dot{}$, $\sqrt{aa-xx}\dot{}$.

Newton explained his notation again in his Quadratura curvarum, London, 1704, where he gave $\overset{3}{x}, \overset{2}{x}, x, \dot{x}, \ddot{x}, \dddot{x}$, each of these terms being the fluxion of the one preceding, and the fluent of the one that follows. The $\overset{2}{x}$ and $\overset{3}{x}$ are fluent notations. His notation for the fluxions of fractions and radicals did not meet with much favor because of the typographical difficulties. In another place‡ he writes "$\dot{z} : \dot{x}$," which in modern notation would be $\frac{dz}{dt} : \frac{dx}{dt}$, or the derivative $\frac{dz}{dx}$.

English writers of the eighteenth century used the Newtonian dots almost exclusively, but these symbols are found also on the European Continent. Thus the Frenchman, Alexis Fontaine,§ uses $\dot{x}$, $\dot{y}$, $\dot{s}$ without defining the notation except by $\dot{x} - x = \dot{x}$, $\dot{y} - y = \dot{y}$, where x, y and $\dot{x}, \dot{y}$ are points on a curve, not far apart, and $\dot{s}$ is the arc from (x, y) to $(\dot{x}, \dot{y})$. The symbols are not used as velocities, but appear to be small constant or variable increments or variations, which may be used by the side of the Leibnizian differential "d," thus $d\dot{s} = \frac{\dot{y}\,dy}{\dot{s}}$. In deriving the curve of quickest descent, he ends with the statement "*Fluxion* $\frac{\dot{y}}{x^{\frac{1}{2}}\dot{s}} = 0$, dont la *Fluente* est $\frac{\dot{y}}{x^{\frac{1}{2}}\dot{s}} = \frac{1}{a^{\frac{1}{2}}}$, que l'on trouvera être l'équation de la cycloide."

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* S. P. Rigaud, Historical Essay on the first Publication of Sir Isaac Newton's Principia, Oxford, 1838, Appendix, p. 23. Consult the remarks on this passage made by G. Enestrom in Bibliotheca mathematica, 3. F., Bd. 11, Leipzig, 1910–1911, p. 276, and Bd. 12, 1911–1912, p. 268, and by A. Witting in Bd. 12, p. 56–60. See also A Catalogue of the Portsmouth Collection of Books and Papers, written by or belonging to Isaac Newton, Cambridge, 1888, p. XVIII, 2. It is of some interest that in 1668, before Newton's notation had appeared in print, N. Mercator used the dot over the letter I, thus $\dot{\text{I}}$, to signify an infinitesimal, and over numbers, $\dot{6}\dot{4}$ for example, to serve as a reminder that 64 was the coefficient of a power of an infinitesimal. See Phil. Trans., London, vol. 3, p. 759 ff. Reprinted in Maseres, Scriptores Logarithmici, vol. 1, p. 231. See also Nature, vol. 109, 1922, p. 477.

† J. Wallis, De Algebra Tractatus, Oxoniae, 1693, p. 390–396.

‡ Isaaci Newtoni Opera (Edition S. Horsley), London, vol. 1, 1779, p. 410.

§ Mémoires donnés à l'académie royale des sciences, non imprimés dans leur temps. Par M. Fontaine, de cette Académie, Paris, 1764, p. 3. Fontaine uses the dot also in articles in Histoire de l'académie r. des sciences, année 1734, Paris, 1736, mémoires p. 372; année 1767, Paris, 1770, p. 588, 596. For further data, see F. Cajori, "Spread of Calculus Notations," Bulletin Am. Math. Soc., vol. 27, 1921, p. 455.

In a paper dated 1748 Fontaine* treats differential equations and uses $\dot{x}, \dot{y}$ in place of dx, dy. Differentiating $n(1-n)p+2nx+(2-5n)y=0$, he obtains $2n\dot{x}+(2-5n)\dot{y}=0$; eliminating n, he arrives at "l'équation aux premières différences $3p\dot{y}^2+10x\dot{y}^2-10y\dot{x}\dot{y}-2p\dot{x}\dot{y}-4x\dot{x}\dot{y}+4y\dot{x}^2=0$." The Newtonian dot was used also by a few other Continental writers† of the eighteenth century and were found regularly in the republications on the Continent of the works of Newton and of a few other Englishmen.

G. W. Leibniz. The earliest use by Leibniz of a calculus symbol was in a letter,‡ dated October 26, 1675, in which he designated by ω the differential of the variable y. Three days later§ he wrote l or $\frac{y}{d}$ in place of ω. On November 11, 1675, he wrote in a manuscript,‖ at first $\frac{x}{d}$ as the differential of x; then he introduced definitely the symbols dx and dy as the differentials of x and y, respectively. In that document he uses also the derivative form $\frac{dx}{dy}$.

The first appearance of dx in print was in an article which Leibniz contributed to the Acta Eruditorum, in 1684.¶ Therein occur the expressions dw ad dx, dx ad dy and also $dx:dy$, but not the form $\frac{dx}{dy}$.

Leibniz used the second derivative** written in the form $ddx:dy^2$. In his Historia et origo calculi differentialis, written by Leibniz not long before his death, he lets†† $dx=1$, and writes $d(x^3)=3xx+3x+1$. In his paper,

* Op. cit., p. 85.

† For further details, see F. Cajori, loc. cit. In addition, I cite the following adherents to the Newtonian notation: P. Steenstra in his Verhandeling over de Klootsche Driehoeks-Meeting, Amsterdam, 1770, p. 167; A. B. Strabbe, Erste beginselen der fluxie-rekening, Amsterdam, 1799.

‡ C. I. Gerhardt, Briefwechsel von G. W. Leibniz mit Mathematikern, Berlin, 1899, p. 150.

§ Loc. cit., p. 153, 155.

‖ Loc. cit., p. 162–164. Also C. I. Gerhardt, Leibnizens Mathematische Schriften, Zweite Abth., Band I, Halle, 1858 (Werke, vol. 5), p. 217. Hereafter we quote this as "vol. 5." Leibniz's representation of the differential of y successively by ω, l, $\frac{y}{d}$, and finally by dy goes against the conjecture that he might have received a suggestion for the notation dy, dx, from Antoine Arnauld who, in his Nouveaux Elémens de Géométrie, Paris, 1667, adopts $d'x$, $d'y$ as the designation of a small, aliquot part of x and y. "Could this French $d'x$ have been hovering before Leibniz when he made his choice of notation?" is the question which naturally arose in the mind of Karl Bopp in Abhandlungen zur Geschichte der mathematischen Wissenschaften, vol. 14, Leipzig, 1902, p. 252.

¶ See also Leibnizens Mathematische Schriften, vol. 5, Halle, 1858, p. 220.

** Acta Eruditorum of 1693, p. 179. Leibnizens Mathematische Schriften, vol. 5, 1858, p. 287.

†† Leibnizens Mathematische Schriften, vol. 5, Halle 1858, p. 406.

printed in 1684, dx appears to be finite when he says: "We now call any line selected at random dx" (Iam recta aliqua pro arbitrio assumta vocetur dx).*

Leibniz writes $(dx)^2$ in the form $dx\,dx$; J. M. Child remarks,† "Leibniz does not give us an opportunity of seeing how he would have written the equivalent of $dx\,dx\,dx$; whether as dx^3 or $\overline{dx}^3$ or $(dx)^3$." In the case of a differential of a radical, Leibniz uses a comma after the d, thus‡ $d, \sqrt[b]{x^a}$, *aeq.* $\frac{a}{b}\,dx\sqrt[b]{x^{a-b}}$. In imitation of Leibniz's ddv for the second differential, Johann Bernoulli§ denotes the fourth differential by $dddd\,z$. Not till 1695, or twenty years after Leibniz's first use of d, did he suggest the use of numbers in the writing of the higher differentials; in a letter‖ of October 20/30, 1695, addressed to Johann Bernoulli, he writes the equation showing the relationship between the two symbols, $d^m = \int^n$ when $n = -m$.

In the same year he published a reply to B. Nieuwentijt's¶ attacks upon the calculus, in which he writes ddx seu $d^2 \cdot x$, $dddx$ sive $d^3 \cdot x$.

De l'Hospital,** who was the earliest to write a text book on the calculus, followed Leibniz's original practice and did not use numerals in the designation of the higher differentials; he wrote ddy, $dddy$, $ddddy$. As late as 1764 we find these very notations in the writings of Alexis Fontaine.†† William Hales,‡‡ an adherent of Newton's fluxions, in 1804, cites the symbols dx, ddx, $dddx$, $ddddx$ and then declares them to be "minùs elegantèr" than the fluxionary symbols.

Minute adjustments of notation arose in the 18th century. Thus, Euler wrote $d \cdot x\,dy\,dz$ for $d(x\,dy\,dz)$. The same practice is observed by S. F. Lacroix; but $dx\,dy\,dz$ always meant the product of the three differentials. Bézout§§ points out that dx^2 means $(dx)^2$, and $d(x^2)$ means the "différentielle de x^2."

* For the various explanations of the infinitesimal given by Leibniz at different times of his career, see G. Vivanti "Il Concetto d'Infinitesimo" in Giornale di Matematiche di Battaglini, vol. 38 and 39.

† J. M. Child, Early Mathematical Manuscripts of Leibniz, Chicago and London, 1920, p. 55.

‡ Acta Eruditorum, 1684, p. 469. Leibnizens Math. Schriften, vol. 5, Halle, 1858, p. 222.

§ Acta Eruditorum, 1694, p. 439; Joh. Bernoulli Opera, vol. 1, p. 125–128.

‖ Leibnizens Math. Schriften, vol. 3, 1855, p. 221.

¶ Acta Eruditorum, 1695, p. 310; Leibnizens Math. Schriften, vol. 5, p. 321.

** De l'Hospital, Analyse des infiniment petits, 2e édition, Paris, 1715, p. 55. The date of the first edition was 1696.

†† A. Fontaine, Mémoires donnés à l'académie r. des sciences, non imprimés dans leur temps. 1764, p. 73.

‡‡ Maseres, Scriptores Logarithmici, vol. 5, London, 1804, p. 131.

§§ Etienne Bézout, Cours de Mathématiques a l'usage du corps de l'artillerie, Tome 3, Paris, 1797, p. 18.

Lacroix explains* that d^2y^2 is the same thing as $(d^2y)^2$ and in general, $d^n y^m$ indicates $(d^n y)^m$, that $d \cdot x^2$ means the differential of x^2 and $d \cdot uv$ of uv; the point following the d signifying that the operation applies to all that immediately follows.

No substantial changes in the notation for total differentiation were made until after the middle of the eighteenth century. The changes that came were due in most cases to new conceptions of the fundamental operations, conceptions which seemed to demand new symbols.

J. Landen. One of the first in the field was John Landen who published at London in 1758 his Discourse concerning Residual Analysis and in 1764 his Residual Analysis. These publications mark an attempt at arithmetization of the calculus, but the process was so complicated as to be prohibitive. In his Mathematical Lucrubations of 1755, Landen still follows Newtonian procedure and notation, but his Discourse of 1758 marks a departure. Still proceeding geometrically, he introduces (p. 12) two new symbols, "$[x\,|\,y]$ being put for the quotient of $y - u$ divided by $x - v$," i.e. for $\Delta y : \Delta x$. Then he, in fact, proceeds to the limit, "by taking v equal to x, and writing $[x \perp y]$ for the value of $[x\,|\,y]$ in the particular case when v is so taken." This notation $[x \perp y]$ for the first derivative labors under the disadvantage of not being compact and requiring many distinct strokes of the pen, but typographically it enjoys an advantage over $\frac{dy}{dx}$ in not requiring a fractional line and in calling for type of normal height. In considering maxima and minima, on p. 24, Landen invents a new designation, "writing $[x \perp\!\!\!\perp y]$ for the value of the quotient of $[x \perp y] - [v \perp u]$ divided by $x - v$, when v is therein taken equal to x." Landen had no followers in the use of his symbols.

A. Fontaine. In France, Alexis Fontaine introduced a notation for partial differentiation which we shall notice later; he takes $\frac{du}{dx}$ to represent a partial derivative and $\frac{1}{dx}du$ a complete derivative.† This form of the complete derivative is seldom found later. We have seen a trace of it in Da Cunha‡, who in 1790 writes the differential $d\left(\frac{1}{dx}d\frac{dy}{dx}\right)$ and in A. S. de Montferrier§, who

* S. F. Lacroix, Traité du Calcul Différentiel et du Calcul Intégral, Tome I, 2d Ed., Paris, 1810, p. 208; also his Traité élémentaire de calcul différentiel et de calcul intégral, Paris, 1802, p. 9, 18.

† A. Fontaine, Mémoires donnés à l'académie r. des sciences, non imprimés dans leur temps. Paris, 1764, in "Table" of the mémoires.

‡ J. A. Da Cunha, Principios mathematicos, Lisbon, 1790, p. 244.

§ A. S. Montferrier, Dictionnaire des sciences mathématiques, Paris, 1835, Art. "Différence", p. 460.

in 1835 writes the derivative $\frac{1}{d\Phi\dot{x}} \cdot d\left[\frac{dF\dot{x}}{d\Phi\dot{x}}\right]$. Here the point placed above the x is to indicate that, after the differentiations, the x is assigned the value which makes $\Phi x = 0$.

J. L. Lagrange. A strikingly new treatment of the fundamental conceptions of the calculus is exhibited in J. L. Lagrange's Théorie des fonctions analytiques, 1797. Not satisfied with Leibniz's infinitely small quantities, nor with Euler's presentation of dx as 0, nor with Newton's prime and ultimate ratios which are ratios of quantities at the instant when they cease to be quantities, Lagrange proceeded to search for a new foundation for the calculus in the processes of ordinary algebra. Before this time the derivative was seldom used on the European Continent; the differential held almost complete sway. It was Lagrange who, avoiding infinitesimals, brought the derivative into a supreme position. Likewise, he stressed the notion of a function. On p. 2, he writes, "Ainsi fx désignera une fonction de x." On p. 14 he proceeds to a new notation for the first, second, third, etc. derivatives of fx, with respect to x. He says: ". . . pour plus de simplicité et d'uniformité, on dénote par $f'x$ la première fonction dérivée de fx, par $f''x$ la première fonction dérivée de $f'x$, par $f'''x$ la première fonction dérivée de $f''x$, et ainsi de suite." "Nous appellerons la fonction fx, *fonction primitive*, par rapport aux fonctions $f'x$, $f''x$, etc., qui en dérivent, et nous appellerons celles-ci, fonctions dérivées, par rapport à celle-là. Nous nommerons de plus la première fonction dérivée $f'x$, *fonction prime*; la seconde dérivée $f''x$, *fonction seconde*; la troisième fonction dérivée $f'''x$, *fonction tierce*, et ainsi de suite. De la même manière, si y est supposé une fonction de x, nous dénoterons ses fonctions dérivées par y', y'', y''', etc."

The same notation for the total derivatives of fx was used by Lagrange in his Leçons sur le calcul des fonctions which appeared at Paris in 1801 and again in 1806.*

It should be noticed that the 1797 edition of Lagrange's Théorie des fonctions is not the first occurrence of the accent as the mark for derivatives. In 1770 Ψ' occurs for $\frac{d\Psi}{dx}$ in his Nouvelle méthode pour résoudre les équations littérales and in 1759, the notation is found in a part of a memoir by François Daviet de Foncenex in the Miscellanea Taurinensia, believed to have been written for Foncenex by Lagrange himself.†

* Reprints are not always faithful in reproducing notations. Thus, in the 1806 edition of the *Leçons*, p. 10, 12, one finds fx, $f'x$, $f''x$, $f'''x$, while in the *Œuvres*, vol. 10, which purports to be a reprint of the 1806 edition, on p. 15, 17, one finds the corresponding parts given as $f(x)$, $f'(x)$, $f''(x)$, $f'''(x)$.

† P. E. B. Jourdain in Proceed. 5th Intern. Congress, Cambridge, 1913, vol. II, p. 540.

J. Pasquich. Another writer who at this time attempted a reform of the calculus was Johann Pasquich of Ofen (Budapest) in Hungary, in a paper entitled, *Exponential Rechnung.** Postulating that every function can be expressed in the form $y = Ax^a + Bx^b + \cdots$, he calls the function $\varepsilon y = aAx^a + bBx^b + \cdots$ the *exponential* of y; "εy" is really the limit of $\frac{x\Delta y}{\Delta x}$.

J. P. Grüson. A change similar to that of Pasquich was suggested by Johann Philipp Grüson of Berlin in his *Calcul d'exposition.*† He reaches the same limit which had been used by Pasquich. Grüson indicated it by an inverted and rounded E; thus $\ni F$ represented the limit of $\frac{x\Delta F}{\Delta x}$. This notation is mentioned by S. F. Lacroix in his Traité du Calcul différentiel et du Calcul intégral.

The notations of Pasquich and Grüson found no acceptance among the mathematicians of their time and have not been drawn upon since for the designation originally assigned. The notation of Lagrange, on the other hand, was adopted by many and has been found a useful adjunct of other calculus notations.

L. F. A. Arbogast. Still another symbolism, which has been extensively used in recent years, was made public by L. F. A. Arbogast in his De Calcul des Dérivations, Strasbourg, 1800. This new calculus was offered as comprising the theory of series, and included the differential calculus as a special case. An outline of it was presented in the form of a memoir in 1789 to the Academy of Sciences at Paris, but was not published at the time. It was mentioned in the 1797 edition of Lagrange's Théorie des fonctions analytique as well as in the first volume of S. F. Lacroix's Traité du Calcul différentiel et du Calcul intégral. Arbogast expresses himself in his Preface as follows:

"To form the algorithm of derivations, it became necessary to introduce new signs; I have given this subject particular attention, being persuaded that the secret of the power of analysis consists in the happy choice and use of signs, simple and characteristic of the things which they are to represent. In this regard I have set myself the following rules: (1) To make the notations as much as possible analogous to the received notations; (2) Not to introduce notations which are not needed and which I can replace without confusion by those already in use; (3) To select very simple ones, yet such that will exhibit all the varieties which the different operations require."

His principal symbol is the D, as the sign for "derivation". During the latter part of the eighteenth century the D had been used by several authors

* J. Pasquich, "Anfangsgründe einer neuen Exponentialrechnung" in Archiv der reinen und angew. Math., II, 1798, p. 385–424. Our information is drawn from M. Cantor, op. cit., vol. IV, p. 667, 668.

† J. P. Grüson, "Le Calcul d'exposition," Mém. académie, Berlin, 1798 (Publ. 1801), p. 151–216; 1799 and 1800 (publ. 1803), p. 157–188.

to represent a finite difference. Arbogast lets $F(a+x)$ be any function (une fonction quelconque) of the binomial $a+x$; one knows, he says, that one can develop that function in a series proceeding according to the powers of x, viz., $a+bx+\frac{c}{1\cdot 2}x^2+\cdots$, where $a=Fa$. He designates by D the operation upon Fa that yields b, so that $b=DFa$, $c=DDFa$, etc. While Arbogast's symbol D for our derivative has maintained its place in many books to the present time, a large variety of satellites to it, which Arbogast introduced, are now only of antiquarian interest. Placing a dot (p. 2) after the D gives him $D\cdot Fa$ to represent $DFa\cdot D\cdot a$, for cases where $D\cdot a$ is not 1. He writes (p. 33) "$\overset{c}{D}{}^m$ au lieu de $\frac{D^m}{1\cdot 2\cdot 3\ldots m}$." On p. 308 he states: "Nous désignerons même à l'avenir les coëfficiens différentiels $\frac{dqx}{dx}, \frac{d^2qx}{1\cdot 2\cdot dx^2}, \ldots$, dx étant invariable, par $\overset{c}{\partial} qx$, $\overset{c}{\partial}{}^2 qx, \ldots$, ce qui est la même chose que les dérivées $\overset{c}{D}qx$, $\overset{c}{D}{}^2 qx, \ldots$."

On p. 330 Arbogast explains d^{-1}, $d^{-2}, \ldots$ and D^{-1}, $D^{-2}, \ldots$ as meaning respectively différentielles inverses and dérivées inverses. As an incidental rather than a systematic notation, the d^{-1} and D^{-1} have maintained themselves to the present day. This notation was used, for instance, by Georg Tralles* in 1804, by A. R. Forsyth in 1903.†

C. Kramp. Arbogast's symbol for derivatives was adopted by C. Kramp, professor of mathematics and physics at Cologne, in his Éléments D'Arithmétique universelle, Cologne, 1808, p. 265, 271. He takes $X=Ax^a+Bx^b+Cx^c+$ etc. and then designates by DX "ce que devient cette fonction, lorsqu'on multiplie tous ses termes par leurs exposans respectifs, et qu'ensuite on divise par x." Similarly for D^2X, etc. With great faith in the combinatorial analysis then flourishing in Germany under the leadership of Hindenburg, Kramp says (p. X): "Arbogast is obliged to assume the theorem of Taylor, as well as the ordinary operations of the differential calculus, as perfectly (parfaitement) known and demonstrated. My derivatives on the contrary are perfectly independent of all notion of the infinitely little or of the limit, of the differential or of differences; and is equally disinclined (et bien loin) to be established on the theorem of Taylor, this theorem does not appear but as a simple corollary of a proposition much more general." The notation D is used in connection with the coefficients arising in his general polynomial development.

b. *Criticisms of Eighteenth Century Notations.*

R. Woodhouse. A critical comparison of the Newtonian and Leibnizian notations was made in 1803 by Robert Woodhouse‡ of Cambridge in England.

* G. Tralles, Abhandlungen Berliner Akademie, Math. Klasse, 1804–1811, p. 214 ff.

† A. R. Forsyth, Treatise on Differential Equations, 2d ed., 1888, p. 43; 3d ed., 1903, p. 55.

‡ R. Woodhouse, Principles of Analytical Calculations, Cambridge, 1803, p. XXVII.

He states: "In the simplest cases, perhaps, there is not much exercise of choice, and $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $\dot{x}^2$, $\dot{x}^3$, $\dot{x}^n$ are as neat as dx, d^2x, d^1x, dx^2, dx^3, dx^n. Take a case from Waring p. 299, Meditationes Analyticae,

$$P\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^n}\right)+Q\left(\frac{\overset{n}{\dot{V}}}{\dot{x}^{n-1}\,\dot{y}}\right)+\text{etc.};$$ this by differential

notation is, $P\cdot\dfrac{d^nV}{dx^n}+Q\cdot\dfrac{d^nV}{dx^{n-1}\,dy}+\text{etc.}$

in which it appears to me, that $\dot{x}^n$ is not so clearly distinguished from $\overset{n}{\dot{x}}$ as dx^n is from d^nx.

"Again, $\dddot{xy}$ or $(xy)'''$ is not so convenient as, $d^3(xy)$.

"Again, suppose x, x', x'', etc., to represent successive values of x, then according to the fluxionary notation, the first, second, etc., fluxions are $\dot{x}$, $\ddot{x'}$, $\dddot{x''}$; by differentials, dx, d^2x', d^3x''. Which notation has here the advantage, must be determined by inspection; and if the advantage is asserted to be with the fluxionary, it is impossible to state in words any irrefragable arguments to the contrary.

"I put down a few more instances, $P\overset{r}{\dot{y}}+H\overset{r-1}{\dot{y}}+h\overset{r-2}{\dot{y}}$, (Waring, p. 206, Meditat. Analyt.) $Pd^ry+Hd^{r-1}y+hd^{r-2}y$; in the fluxionary notation the position of the symbols r, $r-1$, $r-2$, introduces ambiguity, since they may be mistaken for indices of P, H, h;

$$\left(\frac{\dot{x}}{\dot{z}}\right)^{\cdot} \qquad (fx)^{\cdot}, \qquad \frac{\dddot{u}}{\dot{x}^2\,\dot{y}},$$

$$d\left(\frac{dx}{dy}\right) \qquad d.fx, \qquad \frac{d^3u}{dx^2\,dy},\ \ldots$$

"The advantage on the side of the differential notation is not, it may be said, very manifest in these examples, and perhaps I had adhered to the notation most familiar to me, had not stronger reasons than what are contained in the preceding cases presented themselves, for adopting letters instead of dots as the significant symbols of operations. These reasons in a few words are: first, in the fluxionary notation, there is no simple mode of expressing the fluxionary or differential coefficients, that affect the terms of an expanded expression; thus, the form for the binomial without putting down the numeral coefficients, by fluxionary notation is,

$$(x+i)^m = x^m+\frac{\dot{x^m}}{\dot{x}}\,i+\frac{\ddot{x^m}}{1\cdot2\,\dot{x}^2}\,i^2+\text{etc.},$$

an awkward mode of expression certainly; and even in the differential notation, the coefficients cannot be expressed, except by fractions, thus,

$$(x+i)^m = x^m + \frac{d(x^m)}{dx}\, i + \frac{d^2(x^m)}{1\cdot 2\cdot dx^2}\, i^2 + \text{etc.},$$

but then, by a slight alteration of this notation, a very commodious one is obtained; thus, using the small capital D to denote the differential coefficient, that is, putting Dx^n for $\frac{d(x^n)}{dx}$, etc., we have,

$$(x+i)^m = x^m + Dx^m \cdot i + \frac{D^2 x^m}{1\cdot 2}\cdot i^2 + \text{etc. } \ldots.$$

We have the elements of a very clear and symmetrical notation, when

Δ denotes the entire difference,
d the differential, or part of the entire difference,
D the differential coefficient,
δ the variation."

S. F. Lacroix. Calculus notations are discussed by S. F. Lacroix,* who says in part: "Euler† stresses ... the defects of the English notation, which becomes difficult to write and also to perceive when the number of points exceeds three, and one does not see how it is possible to indicate it in figures without danger of confusion with exponents; moreover, we know that it is easy to omit a point in writing or that the point may fail to show in the print. Many of these objections apply to the notation of Lagrange, which has besides, the great inconvenience of depriving analysts of the right to represent by the same letter, differently marked with accents, such quantities as have analogous signification. The use of d is not subject to these difficulties; this character is most conspicuous, especially if, considering it as a sign of operation, we

* S. F. Lacroix, Traité du Calcul différentiel, et du calcul intégral, 2. édition, Tome I, Paris 1810, p. 246.

† L. Euler says in his Institutiones calculi differentialis, Petrograd, 1755, p. 100, 101: "This mode of symbolizing, as a matter of fact, cannot be disapproved, when the number of points is small, since it can be recognized instantly by counting; however, if many points are to be written, it carries with it the greatest confusion and very many inconveniences. The tenth differential or fluxion is represented in this exceedingly inconvenient manner $\dddot{\dddot{y}}$, while by our mode of symbolizing, $d^{10}y$, it is most easily recognized. However, if a case arises in which much higher orders of differentials, perhaps even indefinite ones, are to be expressed, then the English notation is plainly unsuitable."

write it in Roman type, and reserve the Italic letter for the designation of magnitudes, a practice which Euler did not follow: finally the numerals applied to the character d, show very clearly the number of times the differentiation has been repeated." Lacroix states that eight symbols have been proposed for the (total) first derivative, the $\frac{dy}{dx}$ of Leibniz, the $\frac{\dot{y}}{\dot{x}}$ of Newton, the $[x \perp y]$ of Landen, the $f'(x)$ and $\frac{y'}{x'}$ of Lagrange, the εy of Pasquich, the $\frac{\Im y}{x}$ of Grüson, and the Dy of Arbogast. Lacroix takes the question of notations seriously (p. 248): "It is a principle avowed by all, that current notations should not be changed except when they are in manifest conflict with the concepts they are intended to represent, or when one can greatly shorten them, or finally when in modifying them, one renders them fit to develop new relations which cannot be exhibited without them. The signs of the differential calculus do not come under any of these cases: the enrichments which Mr. Lagrange has given to Analysis in his Théorie des fonctions, in his Résolution des équations numériques in his Leçons sur le Calcul des fonctions may be represented with equal simplicity and elegance by the usual characters, as one can see in the first edition of this Treatise." Lacroix adds: "Before multiplying the signs in analysis that are already so very numerous that one may well stop to consider the embarrassment coming to those, who in studying and endeavouring to encompass the whole, must continually bring together formulas and operations that are analogous, yet are expressed in different characters." One should strive to "defend those (symbols) in which the Mécanique analytique and the Mécanique céleste are written."

J. L. Lagrange. And what was Lagrange's attitude toward his own new symbolism?* In the autumn of his busy years his openmindedness is splendidly displayed in his attitude taken in the preface to the second edition of his Mécanique Analitique (the spelling of the title in more recent editions is "Mécanique analytique"), where he retains the Leibnizian notation. He says:

"The ordinary notation of the differential calculus has been retained, because it answers to the system of the infinitely small, adopted in this treatise. When one has well comprehended the spirit of this system and has convinced himself of the exactitude of its results by the geometrical method of prime and ultimate ratios, or by the analytical method of derived functions, one may employ the infinitely little as an instrument that is safe and convenient for abbreviating and simplifying demonstrations. It is in this manner that one shortens the demonstrations of the ancients by the method of indivisibles."

Nor had Lagrange any prejudice against the Newtonian dots. He used them in his Leçons sur le calcul des fonctions, 1806, p. 442, for indicating, "dérivées

* J. L. Lagrange, Mécanique Analytique in Œuvres, Tome 11, Paris p. XIV.

par rapport à i," and in the second edition of his Mécanique Analitique to indicate time-derivatives. In the opinion of Thomson and Tait, "Lagrange has combined the two notations with admirable skill and taste."*

c. *Total Differentiation during the Nineteenth Century.*

Barlow and Mitchell. A slight advantage of Lagrange's strokes over the Newtonian dots was that the former were not placed directly above the letter affected, but in the place where exponents are written. It is worthy of observation that at the opening of the nineteenth century a few English writers attempted to improve Newton's notation, in the case of fluxions of higher orders, by placing only one dot above the letter and then indicating the order by a number placed where exponents usually stand. Thus, Peter Barlow† writes $\dot{x}^n$ for the nth fluxion, but $\dot{x}^2$ for $(\dot{x})^2$. James Mitchell,‡ to denote the fluxions of a radical or fraction, places it in a parenthesis and writes the dot in the place for exponents. Sometimes, however, Mitchell and other authors write the fluxions of compound expressions by placing the letter F or f before them. This practice is, of course, objectionable, on account of the danger of misinterpreting the F or f as meaning, not "fluxion," but "function." We must remark also that this use of F or f was not new at this time; but is found much earlier in the writings of George Cheyne§ in London (who also employed Φ to designate "fluxion of") and of A. Fontaine ‖ in France.

Herschel, Peacock and Babbage. The adoption of the Leibnizian notation in England, advocated in 1802 by Woodhouse, was finally secured through the efforts of J. F. W. Herschel, G. Peacock and C. Babbage at Cambridge. Says J. W. L. Glaisher:¶

"Peacock was Moderator in 1817, and he ventured to introduce the symbol of differentiation into the examination, his colleague, however, retaining the old fluxional notation. The old system made its appearance once more in 1818, but in 1819 Peacock was again Moderator, and, having a colleague who shared his views, the change was fully accomplished. The introduction of the notation and language of the differential calculus into the Senate House examination forms an important landmark in the history of Cambridge mathematics."

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* Thomson and Tait, Treatise on Natural Philosophy, new Ed., Cambridge, 1879, p. 158.

† P. Barlow, Dictionary of Pure and Mixed Mathematics, London, 1814, Art. "Differential."

‡ James Mitchell, Dictionary of the Mathematical and Physical Sciences, London, 1823, Art. "Fluxion."

§ Fluentium methodus inversa a Georgio Cheynaeo, M. D., Londini, 1703, p. 1 of the part against A. Pitcairne.

‖ Fontaine in Histoire de l'académie r. des sciences, année 1767, Paris, 1770, p. 589ff.

¶ J. W. L. Glaisher's Presidential Address, Proceedings London Mathematical Society, vol. 18, 1886–87, London, p. 18.

A. L. Crelle. Influenced by Lagrange's Théorie des fonctions analytiques, Crelle bases his calculus* upon Taylor's Theorem. His various notations may be displayed by the different ways of writing that theorem, as shown in Art. 48, p. 144:

"Die Anwendung der verschiedenen .... vorgeschlagenen Bezeichnungen auf die allgemeine Entwicklung von $f(X+k)$ würde in der Zusammenstellung folgende sein:

$$f(X+k) = |X+k| = fX + k\,dfX + \frac{k^2}{2}\,d^2fX \cdots + \frac{k^n}{1\cdot 2\cdots n}\,d^nfX$$

$$= |X| + k\,d|X| + \frac{k^2}{2}\,d^2|X| \cdots + \frac{k^n}{1\cdot 2\cdots n}\,d^n|X|, \text{ oder wenn } fX = Y \text{ heisst,}$$

$$= Y + k\,dY + \frac{k^2}{2}\,d^2Y \cdots + \frac{k^n}{1\cdot 2\cdots n}\,d^nY. \text{ Ferner:}$$

$$f(X+k) = fX + DfX + \tfrac{1}{2}D^2fX \cdots + \frac{1}{1\cdot 2\cdots n}D^n|X|, \ldots."$$

The variables x, y, z are given in small capitals, which gives the page an odd appearance. It will be noticed that with Crelle, d does not signify "differential"; it signifies "Abgeleitete Grösse," the "derived function" of Lagrange, or the "derivation" of Arbogast. Crelle's notation in partial differentiation and in integration will be given later.

A. L. Cauchy. That Cauchy should be influenced greatly by Lagrange is to be expected. If we examine his published lessons of 1823 on the infinitesimal calculus,† and his lessons of 1829 on the differential calculus, we find that he availed himself of the Leibnizian dx, dy, $\frac{dy}{dx}$ and also of the Lagrangian F' and y' for the first derivatives. In this respect he set a standard which has prevailed widely down to our own day.

M. Ohm. Returning to Germany, we find another attempt to rebuild the calculus notations in the works of Martin Ohm.‡ His symbolism of 1829 is made plain by his mode of writing Taylor's theorem:

$$f(x+h) = f(x) + \partial f(x)\cdot h + \partial^2 f(x)\cdot\frac{h^2}{2!} + \cdots$$

* August Leopold Crelle, Rechnung mit veränderlichen Grössen. Erster Band, Gottingen, 1813.

† Cauchy, Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal, Paris, 1823; also his Leçons sur le calcul différentiel, Paris, 1829. Both are reprinted in his Œuvres, 2. série, vol. 4.

‡ Martin Ohm, Versuch eines vollkommen consequenten Systems der Mathematik, vol. III, Berlin, 1829, p. 53, 67, 80, 94, 102, 103, 107.

One notices here the influence of Arbogast, though Ohm uses ∂ in place of D. Ohm writes also

$$\partial\,(A + B\cdot x^m)_x = m\,B\,x^{m-1}, \qquad \partial\,(a^x)_a = x\cdot a^{x-1}, \qquad \partial\,(a^x)_x = a^x\cdot \log a.$$

If y_x is a function of x, then $(\partial^a y_x)_a$ is what becomes of $\partial^a y_x$ when one writes a in place of x. He introduces also the Leibnizian dx, dy and writes $\dfrac{dy}{dx} = \partial y_x$, $d^2 y = \partial^2 y_x \cdot dx^2$, $\dfrac{d^2 y}{dx^2} = \partial^2 y_x$.

Ohm's notation found some following in Germany. For instance, F. Wolff* in 1845 writes f_x to designate a function of x and $y - y' = \partial f_x\,(x - x')$ as the equation of a line through the point x', y'; Wolff uses also the differential notation dx, dy, $d^2 y$, and the derivative $\dfrac{dx}{ds}$.

Cauchy and Moigno. Meanwhile some new designations introduced by Cauchy in France were clamoring for the right of way. Abbé Moigno† in the Introduction (p. XXI) to his calculus expresses regret that he could not altogether disregard custom and banish from his book "the vague and inconvenient notations $\dfrac{dy}{dx}$, $\dfrac{dy}{dx}dx$, $\dfrac{dz}{dx}$, $\dfrac{dz}{dy}$, $\dfrac{dz}{dx}dx$, $\dfrac{dz}{dy}\dfrac{dy}{dx}dx$, to substitute in their place the more compact notations $y'_x = D_x y$, $d_x y$, $z'_x = D_x z$, $z'_y = D_y z$, $d_x z$, $d_y z$, etc." On page 13 he lets $d_x z$ be a differential and z'_x the derivative, where $z = F(y)$ and $y = f(x)$, so that $d_x z = z'_x\,dx$ and therefore $\dfrac{d_x z}{dx} = \dfrac{d_y z}{dy}\cdot\dfrac{d_x y}{dx}$, but "it is the custom to suppress" the indices x and y and to write $\dfrac{dz}{dx} = \dfrac{dz}{dy}\cdot\dfrac{dy}{dx}$.

B. Peirce. Arbogast's D for derivative was adopted by Benjamin Peirce‡ in his Curves, Functions and Forces, the first volume of which was published in Massachusetts in 1841. The calculus notation used in the United States before 1824 was almost exclusively Newtonian, from 1824 to 1841 almost entirely Leibnizian. In B. Peirce's book, "Differential coefficients are denoted by D, D', etc.; thus $Df\cdot x = \dfrac{df\cdot x}{dx}$, $D^2\cdot f\cdot x = D\cdot D\cdot f\cdot x$, or, since dx is independent of x, $D^2\cdot f\cdot x = \dfrac{d\cdot Df\cdot x}{dx} = \dfrac{d^2 f\cdot x}{dx^2}$. Again, "differentials are denoted by the letters δ, δ' etc., d, d', etc."; Peirce writes $e^x = D^? e^x$, and

* F. Wolff, Lehrbuch der Geometrie. Dritter Theil, Berlin, 1845, p. 178, 202, 278.

† L'Abbé Moigno, Leçons de calcul Différentiel et de calcul Intégral, rédigées d'après les méthodes et les ouvrages publiés ou inédits de M. A.-L. Cauchy, Tome I, Paris, 1840.

‡ Benjamin Peirce in his Elementary Treatise on Curves, Functions, and Forces, new edition, Boston and Cambridge, vol. 1, 1852, p. 179, 182, 193, 203. (First Edition, 1841.)

$D\,\mathrm{tang}^{[-1]}x = Dy = \dfrac{dy}{dx} = (1+x^2)^{-1}$. In vol. II, 1846, p. 15 he lets "also $d_c^n \cdot f \cdot x_0$ be the first differential coefficient of $f \cdot x_0$ which does not vanish," where $f \cdot x_0 \neq 0$ or ∞. Here the d_c meant "differential coefficient."

Since the time of B. Peirce the D has held its place in a few of America's best texts, but even in those it has not been used to the complete exclusion of the Leibnizian symbols. Some authors introduce the Dy, in preference to the $\dfrac{dy}{dx}$, to guard the student against the misconception of regarding the derivative as a fraction. Later in the book they begin to avail themselves of the flexibility afforded by the $\dfrac{dy}{dx}$, which allows an easy passage from derivative to differential, or differential to derivative, by the application of simple algebraic rules.

The more minute examination of the continuity of functions gave rise to the consideration of right-hand and of left-hand derivatives and to corresponding notations. Peano* uses D for derivative, writing $D(f, u, x)$ for the derivative of f in the class u, for the value x of the variable, and $D(f, x+Q_0, x)$ for the "dérivée à droite," and $D(f, x-Q_0, x)$ for the "dérivée a gauche."

G. Peacock. A curious extension of the notion of a derivative $\dfrac{d^n y}{dx^n}$ to the case where n is not a positive integer, say $\dfrac{d^{\frac{1}{2}} y}{dx^{\frac{1}{2}}}$, was considered by Leibniz, Euler, Fourier, Cauchy and Liouville; it received the particular attention of George Peacock,† for the purpose of illustrating his "principle of the permanence of equivalent forms." Following Liouville, Peacock takes for example, $\dfrac{d^n (e^{mx})}{dx^n} = m^n e^{mx}$ and then considers the propriety of assuming this relation as a definition of the operation when n is not necessarily a positive integer. This speculation did not lead to results commensurate with those obtained from the generalization of n in the exponential notation a^n, but it illustrates how a suitable notation is capable of suggesting generalizations which a purely rhetorical exposition would not readily do. In England this generalized differentiation demanded the attention also of D. J. Gregory, P. Kelland, and especially of Oliver Heaviside who made important use of it in electromagnetic theory.‡

* G. Peano, Formulaire mathématique, Tome IV, Turin, 1903, p. 168.

† George Peacock, "Report on the Recent Progress and Present State of Certain Branches of Analysis" in British Association Report for 1833, London, 1834, p. 211-221, 241-249. Peacock gives bibliography.

‡ Proceedings of the Royal Society of London, Part I, Febr. 2, 1893; Part II, June 15, 1893, Part III, June 21, 1894. See also O. Heaviside, Electromagnetic Theory, vol. 2, London, [1899], p. 434, 435.

J. Fourier. Fourier, is his Traité de la chaleur, 1822, showed in connection with the solution of partial differential equations the advantage which may accrue from the separation of the symbols of operation from those of quantity. Thus, in Chapter IX, Section IV: "If the proposed differential equation is

$$\frac{d^2v}{dt^2} + \frac{d^4v}{dx^4} + 2\frac{d^4v}{dx^2dy^2} + \frac{d^4v}{dy^4} = 0$$

we may denote by D_Φ the function $\frac{d^2\Phi}{dx^2} + \frac{d^2\Phi}{dy^2}$, so that DD_Φ or D^2_Φ can be formed by raising the binomial $\left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2}\right)$ to the second degree, and regarding the exponents as orders of differentiation."

This symbolic method was developed further in England by S. S. Greatheed* and D. F. Gregory,† both of Cambridge, by George Boole‡ of Queen's University, Ireland, R. Carmichael§ of Dublin, and by many others in Great Britain and the Continent.

d. *Partial Differentials and Partial Derivatives.*

L. Euler. Partial derivatives appear in the writings of Newton, Leibniz, and the Bernoullis, but as a rule without any special symbolism. To be sure, in 1694, Leibniz,‖ in a letter to De l'Hospital, wrote "δm" for the partial derivative $\frac{\partial m}{\partial x}$ and "ϑm" for $\frac{\partial m}{\partial y}$, and de l'Hospital uses "δm" in his reply of March 2, 1695. In 1728 L. Euler¶ used new letters P, Q, R, to designate the partial derivatives with respect to x, y, z, respectively. Similarly G. Monge** used the small letters p and q as partial derivatives of z with respect to x and y, respectively, and r, s, t as partial derivatives of the second order. In more recent time Monge's abbreviations have been employed by writers on differential equations and differential geometry. Euler indicated in 1755, in his Institutiones calculi differentialis, vol. I, p. 195, the partial derivative of P with respect to y by the use of a parenthesis. Thus, $\left(\frac{dP}{dy}\right)$ signifies that "P is

* S. S. Greatheed, in Philosophical Magazine, Sept., 1837.

† D. F. Gregory in Cambridge Mathematical Journal, vol. I, p. 123.

‡ George Boole, Differential Equations, 4th edition, London, 1877, Chapters 16, 17.

§ Robert Carmichael, Calculus of Operations, London, 1855.

‖ Leibnizens Mathematische Schriften, vol. 2, Berlin, 1850, p. 261, 270.

¶ Commentarii Academiae Scientiarum imperialis, Petropolitanae, Tome 3, (1728), Ed. 1732, p. 115, 116; Encyclopédie des sciences Mathématiques, Tome II, vol. 1, Paris and Leipzig, 1912, p. 284, Note 168.

** Gaspard Monge, Application de l'analyse à la géométrie, edited after the 4th Edition of 1809 by J. Liouville, Paris, 1850, p. 53 and 71.

so differentiated that y alone is treated as a variable, and that differential is divided by dy."*

The notation $\left(\frac{dy}{dx}\right)$ for partial derivatives has been widely used, notwithstanding the objections which have been raised to it. If one writes $\left(\frac{dp}{dx}\right)^2 + \left(\frac{dp}{dy}\right)^2 + \left(\frac{dp}{dz}\right)^2 = 1$ or $a\left(\frac{dy}{dx}\right)$, the reader may be in doubt whether the parentheses are introduced to indicate the presence of partial derivatives or merely to render algebraic service in the expression of the square or the product. These objections to the contrary notwithstanding, this Eulerian notation long held its place in competition with other notations for partial derivatives. Some idea of its wide distribution both in time and localities will be conveyed by the following references: It was used by L'Abbé Sauri,† of Paris in 1774, by O. Gherli‡ in Modena in 1777, by E. Waring§ of Cambridge in 1785, by P. S. Laplace‖ in Paris, in 1772–1798, by G. Monge¶ in 1784, by A. M. Legendre** in 1784, by J. L. Lagrange†† in 1759, 1792, by A. S. de Montferrier‡‡ in Paris in 1835, by B. Price§§ at Oxford, 1852, by Th. Strong‖‖ in New York in 1869, by M. H. Resal¶¶ at Paris in 1884, by S. Newcomb*** of Washington in 1887, and by John Perry††† of London who in 1902 expressed his preference for the Eulerian notation.

* "Brevitatis gratia autem hoc saltem capite quantitates r et q ita commode denotari solent, ut r indicetur per $\left(\frac{dP}{dy}\right)$, quae scriptura designatur P ita differentiari, ut sola y tanquam variabilis tractetur, atque differentiale istud per dy dividatur." Just above that passage Euler wrote "posito ergo x constante erit $dP = r\,dy$, ... Deinde posito y constante erit $dQ = q\,dx$."

† L'Abbé Sauri, Cours Complet de mathématiques, Tome V, Paris, 1774, p. 83.

‡ O. Gherli, Gli Elementi teorico-practii delle matematiche pure, Tomo VII, Modena, 1777, p. 2.

§ E. Waring, Meditationes analyticae, Cambridge, 1785, p. 166.

‖ P. S. Laplace, Traité de mécanique céleste, Tome I, Paris, vol. VII (1798), p. 7. Histoire de l'académie r. des sciences, année 1772, Paris, 1775, mémoires, p. 346.

¶ G. Monge in Histoire de l'académie r. des sciences, année 1784, Paris, 1787, p. 119.

** A. M. Legendre, in loc. cit., année 1784, p. 372.

†† J. L. Lagrange in Mémoires de l'académie r. des sciences et belles-lettres, année 1792 et 1793, Berlin, 1798, p. 302: Miscellanea philosophico-mathematica societatis privatae Taurinensis, T. I, Propagation du son, p. 18.

‡‡ A. S. de Montferrier, Dictionnaire des sciences mathématiques, Tome I, Paris, 1835, Art. "Différence."

§§ B. Price, Treatise on Infinitesimal Calculus, vol. I, Oxford, 1852, p. 118.

‖‖ Theodore Strong, Treatise of the Differential and Integral Calculus, New York, 1869, p. 44.

¶¶ H. Resal, Traité élémentaire de mécanique céleste, 2. Édition, Paris, 1884, p. 8.

*** Simon Newcomb, Elements of the Differential and Integral Calculus, New York, 1887, p. 54.

††† John Perry in Nature, vol. 66, 1902, p. 271.

In 1776 Euler* uses $\frac{\partial^\lambda}{p} \cdot V$ to indicate the λth derivative, partial with respect to the variable p; he uses a corresponding notation $\int\limits_{x}^{s} \cdot V$ for partial integration. A similar notation is adopted by F. Servois† in 1814.

W. J. G. Karsten. The need of some special designation must have been felt by many workers. Apparently unaware of Euler's notation, W. J. G. Karsten‡ in 1760, taking V as a function of x, y, z, u, etc., denotes the partial increment with respect to x by $\overset{x}{\Delta} V$, the partial increment of $\overset{x}{\Delta} V$ with respect to y, by $\overset{y}{\Delta}\overset{x}{\Delta} V$; correspondingly, he writes the partial differentials $\overset{x}{d} V$ and $\overset{y}{d}\overset{x}{d} V$; he designates also the partial differential with respect to y of the partial integral with respect to x, $\overset{y}{d} \int\limits^{x} P dx$. Similarly for partial derivatives, he writes $\frac{\overset{y}{d} P}{dy} = \frac{\overset{x}{d} Q}{dx}$.

This notation did not attract much attention. It involved, in the last illustration, four terraces of type, which was objectionable. Moreover, German writers in Karsten's time were not in the lime-light, and not likely to secure a following outside of their own country.

A. Fontaine. Attention must be paid to the notation in partial differentiation used by Alexis Fontaine§ of Paris, who presented certain memoirs before the French Academy in 1739, but which were not published until 1764. The notation for partial differentials used by him is found in text books even of our own time.

In the "Table" of contents of the memoirs contained in the volume one finds μ given as a function of x, y, z, u, etc., and then

$$d\mu = \frac{d\mu}{dx} dx + \frac{d\mu}{dy} dy + \frac{d\mu}{dz} dz + \frac{d\mu}{du} du + \text{etc.}$$

$$\frac{dd\mu}{dx} = \frac{dd\mu}{dx^2} dx + \frac{dd\mu}{dx\,dy} dy + \frac{dd\mu}{dx\,dz} dz + \frac{dd\mu}{dx\,du} du + \text{etc.}$$

* Euler in Nova acta academiae scientiarum i. Petropolitanae, T. IV, ad annum 1786, Petrograd, 1789, p. 18.

† Servois in Gergonnés Annales de mathématiques, T. V, Nismes, 1814, 1815, p. 94, 95.

‡ W. J. G. Karsten, Mathesis Theoretica Elementaris atque sublimior, Rostochii, 1760, p. 775, 781, 782, 807, 808.

§ Mémoires donnés à l'académie royale des sciences, non imprimés dans leur temps. Par M. Fontaine, de cette académie, Paris, 1764. The part in the book containing definitions, notations and fundamental theorems of the "calcul intégral" on page 24, is dated Nov. 19, 1738.

and similarly expressions for $\frac{dd\mu}{dy}$, $\frac{dd\mu}{dz}$, $\frac{dd\mu}{du}$. Evidently, the $d\mu$ in the left member of the first equation is the total differential of the first order, the $\frac{dd\mu}{dx}$ in the left member of the second equation is the total differential of the partial derivative $\frac{d\mu}{dx}$. The partial differential with x as the sole variable is here indicated by $\frac{d\mu}{dx}\,dx$; with y as the sole variable, by $\frac{d\mu}{dy}\,dy$, and similarly for z, u, etc. This notation lacks compactness, but has the great merit of being easily understood and remembered.

Fontaine's equations in the "Table" are followed by the following explanation:

"Cette expression-ci $\frac{1}{dx}\cdot d\mu$ est donc bien différente de celle-cu $\frac{d\mu}{dx}$. La première signifie la différence de μ divisée par dx; la seconde signifie le coëfficient de dx dans la différence de μ."

In other words, here the $\frac{d\mu}{dx}$ is our modern $\frac{\partial\mu}{\partial x}$. As the only derivatives occurring in Fontaine's equations are partial derivatives, no confusion could arise in the interpretation of derivatives; $\frac{dd\mu}{dx\,dy}\,dy$ meant the partial differential (with respect to y as the only variable) of the partial derivative $\frac{\partial\mu}{\partial x}$, while $\frac{dd\mu}{dx\,dy}$ indicated $\frac{\partial^2\mu}{\partial x\,\partial y}$. This interpretation is confirmed by statements on page 25.

The wide spread of Fontaine's notation for partial differentials may be displayed as follows: It was used by D'Alembert* in 1763, J. L. Lagrange† in 1788, S. D. Poisson‡ in 1818, K. F. Gauss§ in 1829, F. Moigno‖ in 1840, J. M. C. Duhamel¶ in 1874, Ch. Hermite** in 1873, Ch. Davies and W. G. Peck†† in 1876.

* D'Alembert in Mémoires de l'académie, Tome XIX, année 1763, Berlin, 1763, p. 264.

† J. L. Lagrange, Méchanique analitique, 1788, p. 137.

‡ S. D. Poisson in Mémoires de l'académie r. des sciences de l'Institut de France, année 1816, Tome I, Paris, 1818, p. 21.

§ Gauss, Werke, vol. V, Göttingen, 1867, p. 57. Gauss says: "Spectari itaque potererit z tanquam functio indeterminatarum x, y, cuius differentialia partialia secundum morem suetum, sed omissis vinculis, per $\frac{dz}{dx}\cdot dx$, $\frac{dz}{dy}\cdot dy$ denotabimus."

‖ L'Abbé Moigno, Leçons de calcul différentiel et de calcul intégral, Paris, 1840, Tome I, p. 20.

¶ J. M. C. Duhamel, Éléments de calcul infinitésimal, 3. ed., Paris, 1874, p. 273.

** Ch. Hermite, Cours d'analyse, 1. Partie, Paris, 1873, p. 80.

†† Charles Davies and William G. Peck, Mathematical Dictionary, New York and Chicago, 1876, Art. "Partial Differential."

A notation more along the line of Karsten's is found in Abbé Girault* de Koudou's calculus which is a curious mixture of English phraseology and German symbols. We quote: "Cette expression $d^{x}(X)$ signifie Fluxion de X dans la supposition de x seule variable," where $X = x + y + z - u + a$.

Monge, Condorcet and Legendre. About 1770 G. Monge† wrote $\frac{\delta V}{dx}$ and $\frac{dV}{dy}$ to represent the partial derivatives with respect to x and y, respectively. In an article of the year 1770 Marquis de Condorcet‡ indicated a partial differential of z with respect to x by dz, and a partial differential of z with respect to y by ∂z, or else he indicated by dz a total differential and by ∂z a partial. "Jusqu'ici l'une ou l'autre hypothèse donne le méme résultat, mais lorsqu'il en sera autrement, j'aurai soin d'avertir de celle dont il sera question." In an article§ of 1772 he alters the meaning of d and ∂ so as to signify differentiation, with respect to y and x, respectively.

The use of the rounded letter ∂ in the notation for partial differentiation occurs again in 1786 in an article by A. M. Legendre. He introduces the modern notation $\frac{\partial v}{\partial x}$ for the partial derivative and says: "Pour éviter toute ambiguité, je représenterai par $\frac{\partial v}{\partial x}$ le coéfficient de dx dans la différence de v et par $\frac{dv}{dx}$ la différence complette de v divisée par dx."

How did it happen that Legendre's happy suggestion remained unheeded? There were perhaps two reasons. One was that he himself failed to use his notation in later articles. The year following he not only abandoned his own notation, but also that of Euler which he had used in 1784, and employed no symbolism whatever to distinguish between total and partial differentiation. Perhaps a second reason lies in the fact that at that time there was a great scramble for the use of the letters D and d in different fields of mathematics. The straight d had been used by Leibniz and his followers, but writers on finite differences¶ laid claim to it, as well as to the capital D.

* Leçons analytiques du Calcul des Fluxions et des Fluentes, ou Calcul Differentiel et Intégrale. Par M. L'Abbé Girault de Koudou, Paris, MDCCLXXII.

† Monge in Miscellanea Taurinensia, T. V, Part II, 1770–1773, p. 94.

‡ Condorcet in Histoire de l'académie r. des sciences, année 1770, Paris, 1773, p. 152.

§ Condorcet in loc. cit., année 1772, Part I, Paris, 1775, mémoires, p. 14.

‖ Legendre in loc. cit., année 1786, Paris, 1788, p. 8. The article is translated by Stäckel in Ostwald's Klassiker der exakten Wissenschaften, No. 78, Leipzig, 1896, p. 64–65. See also Encyclopédie des sciences mathématiques, Tome II, vol. I, p. 284, Note 168.

¶ Among the writers who used the D in finite differences were L'Abbé Sauri, Cours complet de mathématiques, Tome III, Paris, 1774, p. 304; Georg Vega, Vorlesungen über die Mathematik, vol. II, Wien, 1784, p. 395; J. A. Da Cunha, Principios mathematicos, 1790,

Thereupon Arbogast gave D the new meaning of "derivation." It looked indeed as if the different mathematical architects engaged in erecting a proud mathematical structure found themselves confronted with the curse of having their sign language confounded so that they could the less readily understand each other's speech. At this juncture certain writers on the calculus concluded that the interests of their science could be best promoted by discarding the straight-letter d and introducing the rounded ∂. Accordingly they wrote ∂y for the total differential, $\frac{\partial y}{\partial x}$ for the total derivative, and $\left(\frac{\partial y}{\partial x}\right)$ for the partial derivative. G. S. Klügel refers to this movement when he says in 1803:* "It is necessary to distinguish between the symbol for differential and that for a (finite) quantity by a special form of the letter. In France† and in the more recent memoirs of the Petersburg Academy, writers‡ have begun to designate the differential by the curved ∂." So Klügel himself adopts this symbolism from now on for his dictionary. Euler's Institutiones calculi integralis which in its first edition of 1768–70 used the straight d, appeared in the third edition of 1824 with the round ∂, both for the total differential and the total derivative. This same notation is found in J. A. Grunert's calculus§ of 1837. However, the movement failed of general adoption; for some years both d and ∂ were used by different writers for total differentiation.

In view of these counter currents it is not surprising that Legendre's genial notation for partial differentiation failed of adoption.

J. L. Lagrange. Lagrange, in his Méchanique analitique of 1788, used no special signs for partial differentiation; he even discarded Euler's parentheses. The reader was expected to tell from the context whether a derivative was total or partial. We have seen that this attitude was taken also by Legendre in 1787.

S. F. Lacroix. Lagrange's procedure of 1788 was adopted by S. F. Lacroix whose three-volume treatise on the calculus was a standard work of reference in the early part of the 19th century. Lacroix says:‖ "First, let me observe

p. 269. The small letter d was used less frequently in finite differences; we cite Charles Hutton who in his Mathematical and Philosophical Dictionary, London, 1895, Art. "Differential Method," employs the d.

* G. S. Klügel, Mathematisches Worterbuch, 1. Abth., 1. Theil, Leipzig, 1803, Art. "Differentiale."

† In France Le Marquis de Condorcet had used the ∂ in writing total derivatives in his Probabilité des décisions, in 1785.

‡ The Nova Acta Petropolitana, the first volume of which is for the year 1785, contain the rounded ∂.

§ Johann August Grunert, Elemente der Differential und Integralrechnung, Leipzig, 1837.

‖ S. F. Lacroix, Traité du Calcul Différentiel et du Calcul Intégral, Tome I, 2. Ed., Paris, 1810, p. 243.

that it (the partial derivative) should be disencumbered of the parentheses which Euler employed. Really, $\frac{dz}{dx}$, $\frac{dz}{dy}$ are as clear as $\left(\frac{dz}{dx}\right)$, $\left(\frac{dz}{dy}\right)$ when one knows beforehand that z is a function of two independent variables x and y, which the statement of the question, or the meaning of which it is susceptible, always indicates. ... Messrs. Lagrange and Legendre have long ago suppressed the parentheses in their calculations and I have felt that, without inconvenience, I may follow the example which they have set. Fontaine who was the first to apply the notation of Leibniz to partial differentials, proposed to designate by $\frac{1}{dx}\,d\mu$ the ratio of dx to the total differential of $d\mu$." In a more complicated function of several variables, of say x, y, z, where perhaps z is a function of x and y, Lacroix admits that some special notations should be used such as had been proposed by Lagrange (in 1797 and 1801).

Lacroix's attitude toward partial derivatives has had a very large following everywhere, almost down to the present; no special notation was employed for the ordinary partial derivative. This was the attitude of Poisson,* of Cauchy† (in 1821 and 1822), of J. A. Serret,‡ of J. M. C. Duhamel,§ of Ch. Hermite‖ and others in France. This was the practice of George Boole,¶ Isaac Todhunter,** Lord Rayleigh,†† Thomson and Tait,‡‡ in Great Britain, it being remarked by W. H. Young,§§ as late as 1910, that in English writings the ordinary $\frac{df}{dx}$ was sometimes used to represent the partial derivative. In Sweden a similar course was followed, as is exemplified in papers of A. V. Bäcklund.‖‖

This failure to observe a notational distinction between partial and total derivatives caused the less confusion, because the derivatives were usually obtained from expressions involving differentials, and in the case of differ-

* S. P. Poisson. Traité de mécanique, 2. édition. Tome I, Paris, 1833, p. 36.

† A. Cauchy, Mémoires de l'académie r. des sciences, Tome V, années 1821 et 1822. Paris, 1826, p. 47.

‡ Notes de M. J. A. Serret sur le calcul différentiel et le calcul intégral par Lacroix (no date, but after 1861). But in J. A. Serret's Cours de calcul différentiel et intégral. 2. Ed., Paris, 1879, p. 29, etc., the Legendre-Jacobi notation is used.

§ J. M. C. Duhamel, Éléments de calcul Infinitésimal, 3. Éd., Paris, 1874, p. 252.

‖ Ch. Hermite, Cours d'Analyse, 1. Partie, Paris, 1873, p. 78, 79.

¶ George Boole, Differential Equations.

** I. Todhunter, Treatise on the Differential Calculus, 7th Ed., London, 1875.

†† Baron Rayleigh, The Theory of Sound, London, 1877.

‡‡ Thomson and Tait, Treatise on Natural Philosophy, 1879, p. 160.

§§ W. H. Young, Fundamental Theorems of the Differential Calculus, Cambridge University Press, 1910, p. 20.

‖‖ A. V. Bäcklund in Lunds Univ. Årsskrift, Tom. IX, Lund, 1872, p. 4.

entials some distinctive symbolism was employed—it being either Fontaine's designation or some later notation.

J. A. Da Cunha. Portugal possessed an active mathematician, J. A. Da Cunha,* who in 1790, in a wonderfully compact, one-volume treatise of mathematics used symbolism freely. With him, "d^x denotes the fluxion taken with respect to x, $\int^x$ the fluent taken relative to x." When $M = \frac{d^y O}{dy}$, then $\frac{d^x M}{dx} = \frac{{}'d^2 O}{dy\, dx'}$, the single, inverted commas in the second derivative being used as a reminder that each of the two differentiations is partial.

S. Lhuilier. A notation for partial differentials different from Fontaine's and resembling Karsten's is given in Simone Lhuilier's book on the exposition of the principles of the calculus,† in 1795. When P is a function of x and y, he writes the first three partial differentials with respect to x thus: ${}^x d' P$, ${}^x d'' P$, ${}^x d''' P$; in general, he indicates N partial differentiations with respect to x, followed by M partial differentiations with respect to y, by the differential symbol ${}^y d^M {}^x d^N P$. We shall see that Cauchy's notation of 1823 differs from Lhuilier's simply in the positions of the y and x and in the use of small letters in the place of M and N.

J. L. Lagrange. We return to the great central figure in the field of pure mathematics at the close of the eighteenth century—Lagrange. The new foundations which he endeavoured to establish for the calculus in his Théorie des fonctions analytiques, 1797, carried with it also new notations, for partial differentiation. He writes (p. 92), $f', f'', f''', \ldots$ to designate the first, second, third, etc., functions (derivatives) of $f(x, y)$, relative to x alone; and the symbols $f_1, f_{11}, f_{111}, \ldots$ to designate the first, second, third, ... derivatives, relative to y alone. Accordingly, he writes (p. 93) also $f_1'(x, y)$, $f_1''(x, y)$, $f_{11}'(x, y)$, $f_{11}''(x, y)$, etc., for our modern $\frac{\partial^2 f}{\partial y\, \partial x}$, $\frac{\partial^3 f}{\partial y\, \partial x^2}$, $\frac{\partial^3 f}{\partial y^2\, \partial x}$, $\frac{\partial^4 f}{\partial x^2\, \partial y^2}$, etc. In the case of a function $F(x, y, z)$ of three variables he represents (*Œuvres*, vol. 9, p. 158) by $F'(x)$, $F'(y)$, $F'(z)$ the "primes fonctions" of $F(x, y, z)$ taken with respect to x, y, z, respectively, as independent variables. The danger of this notation is that $F'(x)$ might be taken to be the primitive F' function of the single variable x. Lagrange also (*Œuvres*, vol. 9, p. 177) denotes by u', u_1 and ${}_1u$ the prime functions (first derivatives) of u with respect to x, y, z, respectively. Furthermore (*loc. cit.* p. 273), u'', u_{11}, u_1' stand for our $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 u}{\partial x\, \partial y}$, respectively. In 1841 C. G. J. Jacobi declared that "the notation

* J. A. Da Cunha, Principios mathematicos, Lisboa, 1790, p. 263.

† S. Lhuilier, Principiorum calculi differentialis et integralis expositio elementaris, Tübingen, 1795, p. 238.

of Lagrange fails, if in a function of three or more variables one attempts to write down differentials of higher than the first order."* But P. Stäckel† remarked in 1896 that the Lagrangian defects can be readily removed by the use of several indices, as for example, by writing $f_{\alpha\beta\gamma}$ for our $\frac{\partial^{\alpha+\beta+\gamma} f}{\partial x^{\alpha} \partial y^{\beta} \partial z^{\gamma}}$.

The historical data which we have gathered show that as early as 1760, W. J. G. Karsten had evolved a precise notation even for higher partial derivatives. It was capable of improvement by a more compact placing of the symbols. But this notation remained unnoticed.

That Lagrange was not altogether satisfied with his notation of 1797 appears from his Résolution des équations numériques, 1798, p. 210, where he introduces $\left(\frac{Z'}{a'}\right)$, $\left(\frac{Z'}{b'}\right)$, $\left(\frac{Z''}{a'^2}\right)$, $\left(\frac{Z''}{a'\,b'}\right)$ to designate the partial derivatives $\frac{\partial Z}{\partial a}$, $\frac{\partial Z}{\partial b}$, $\frac{\partial^2 Z}{\partial a^2}$, $\frac{\partial^2 Z}{\partial a\,\partial b}$, and then says: "Cette notation est plus nette et plus expressive que celle que j'ai employée dans la Théorie des fonctions, en plaçant les accents différement, suivant les différentes variables auxquelles ils se rapportent." Lagrange adds the further comment: "In substituting the former in place of this, the algorithm of derived functions conserved all the advantages of the differential calculus, and will have this additional one of disencumbering the formulas of that multitude of d's which lengthen and disfigure them to a considerable extent, and which continually recall to mind the false notion of the infinitely little."

It will be noticed that Lagrange's notation for partial derivatives, as given in 1798 in his Résolution des Équations numérique, and again in his Leçons sur le calcul des fonctions,‡ Paris, 1806, p. 347, closely resembles the notation previously used by Edward Waring at Cambridge. The only difference is that Waring employs the fluxional dot, placed above the letter, as in $\left(\frac{\dot{Z}}{\dot{a}}\right)$, while Lagrange uses a stroke in the position assigned to an exponent, as in $\left(\frac{Z'}{a'}\right)$. It will not escape the reader that in ordinary derivatives as well, the Lagrangian f', f'', f''', bears close resemblance to the Newtonian $\dot{x}$, $\ddot{x}$, $\dddot{x}$, the stroke again replacing the dot. We have previously pointed out that certain English and American writers shifted the Newtonian dots to the position where exponents are placed. By so doing the notations of Newton and Lagrange were brought still closer together.

* C. G. J. Jacobi, "De determinantibus functionalibus", in Journal für d. r. & a. Mathematik, vol. 22, p. 321; Jacobi, Ges. Werke, vol. 3, p. 397.

† P. Stäckel in Ostwald's Klassiker der exakten Wiss., Nr. 78, p. 65.

‡ Lagrange, Œuvres, vol. 10.

L. F. A. Arbogast. In his Calcul des Dérivations, 1800, (p. 89) Arbogast designates by $D^m \cdot \varphi(a, \alpha)$ m total derivations, while $D^{m,n} \cdot \varphi(a, \alpha)$ designates partial derivations, viz., m derivations with respect to a only, followed by n derivations with respect to α only. Accordingly, $D^{o,n} \cdot \varphi(a, \alpha)$ or simply $D^n \varphi(a, \alpha)$ indicates n partial derivations with respect to α. Also $D^{m,n} \cdot \varphi(a, \alpha)$ means that $D^{m,n} \cdot \varphi(a, \alpha)$ is to be divided by $(m!\ n!)$. He uses the Leibnizian d as the sign for the ordinary differential. The symbols d', d' are partial differentials (p. 319). He writes $d' + d'$, $d^n = (d' + d')^n$, $d^n(x, y, z) = (d' + d' + d')^n \times xyz$.

J. L. Lagrange. In close alignment with the Arbogast symbols is the new notation for partial derivatives which Lagrange gave in his Leçons sur le calcul des fonctions, Paris, 1801 (second edition 1806), p. 329, 330, 331. Let $f(x, y)$ be a function of x and y, then $f'(x, y)$ means the partial derivative with respect to x, $f'(x, y)$ the partial derivative with respect to y, $f''(x, y)$ the partial derivative with respect to x, followed by the partial derivative with respect to y. Higher partial derivatives are designated by $f^{m,n}(x, y)$.

Robert Woodhouse referred to these signs in his Principles of Analytical Calculations (Cambridge, 1803), p. XXX, where he compares the English and Continental notations and gives his reasons for adopting the Continental. He says: "Many theorems are conveniently demonstrated by separating the scale of operation, thus, when applied to a rectangle xy, Δ, D, d, may be separated into parts, as Δ', Δ', D', D^d, d', d', where Δ', d', etc. mean respectively the entire difference and differential of xy, relatively to x only; Δ', d', relatively to y only; for instance $d(xy) = x\,dy + y\,dx = d'(xy) + d'(xy) = (d' + d')\,xy$: The advantage resulting from this change of notation, cannot be here distinctly pointed out, but it would be difficult, I apprehend, to introduce a similar and corresponding change in the fluxionary notation; and this is an additional reason, why I made a deviation, for which the English reader may feel a propensity to blame me."

A. L. Crelle. In this restless period in the development of the calculus, there were many minds, and divided councils. We have seen that new notations were suggested by A. L. Crelle.* In partial derivatives, he has a notation resembling Euler's of 1776; Crelle states (p. 72): "When $z = |xy|$, then we let $\frac{d}{x} z$ or $\frac{d}{x} |xy|$, or $\frac{d}{x} fxy$ stand for the first partial derivation according to x of the magnitude $|xy|$, while $\frac{d}{y} z$ or $\frac{d}{y} |xy|$ or $\frac{d}{y} fxy$ stands for the partial derivation according to y, of the magnitude $|xy|$." Similarly, $\frac{d^2}{x^2} z$ or $\frac{d^2}{x^2} fxy$ or $\frac{d^2}{x^2} |xy|$ indicates the second partial derivative with respect to x,

* A. L. Crelle, Rechnung mit veränderlichen Größen, vol. 1, Göttingen, 1813.

while $\frac{d^2}{xy} z$ represents the partial derivative with respect to x, followed by that with respect to y. It is to be noted that with Crelle, d does not signify "differential," but "abgeleitete Größe," which Lagrange called the "derived function" and Arbogast the "derivation." The partial is distinguished from the total derivative, by having the independent variable with respect to which the differentiation takes place, written below the d. Thus dz means the total derivative, $\frac{d}{x} z$ means the partial. On page 82 he introduces also the D for the total derivative.

P. Barlow. A quaint notation was given in England in 1814 by Peter Barlow.* If u is any function of x, y, z, etc., let δ, d, D, etc., be the characteristics of their differentiation relatively to each of the variables, then the partial differences are shown here in the second and third lines,

x,	y,	z,	etc.,
$\frac{\delta u}{dx}$,	$\frac{du}{dy}$,	$\frac{Du}{dz}$,	etc.,
$\frac{\delta\delta u}{dx^2}$,	$\frac{d\delta u}{dx\,dy}$,	$\frac{D\delta u}{dx\,dz}$,	etc.

The association of x, y, z, etc., with δ, d, D, etc., respectively, makes unnecessarily heavy demands upon the memory of the reader.

A. L. Cauchy. Cauchy used a variety of different notations for partial differentiation at different times of his long career. In 1823, in his lessons on the infinitesimal calculus,† he represents partial differentials by $d_x u$, $d_y u$, $d_z u$, and the partial derivatives by $\frac{d_x u}{dx}$, $\frac{d_y u}{dy}$, $\frac{d_z u}{dz}$, but remarks that the latter are usually written, for brevity, $\frac{du}{dx}$, $\frac{du}{dy}$, $\frac{du}{dz}$. He writes also the more general partial derivative $\frac{d_x^l d_y^m d_z^n \cdots u}{dx^l dy^m dz^n \cdots}$ but adds that the letters x, y, z at the base of the d's are ordinarily omitted, and the notation $\frac{d^{l+m+n\cdots} u}{dx^l dy^m dz^n \cdots}$ is used. The same notation, and similar remarks on partial derivatives and partial differentials, are found in his lessons on the differential calculus‡ of 1829.

* P. Barlow, Dictionary of Mathematics, 1814, Art. "Partial Differences."

† Cauchy, Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal, Paris, 1823; Œuvres, II. série, vol. 4, p. 50, 79.

‡ Cauchy, Leçons sur le calcul différentiel, Paris, 1829; Œuvres, II. série, vol. 4, p. 513, 527.

In England John Hind* gives the general form of partial derivatives which we last quoted for Cauchy, and then remarks: "Another kind of notation attended with some conveniences has of late been partially adopted. In this, the differential coefficients are no longer expressed in a fractional form, but are denoted by the letter d with the principal variables suffixed: thus, $\frac{du}{dx}$ and $\frac{du}{dy}$ are equivalent to $d_x u$ and $d_y u$," and so that the above partial derivative of the order $l+m+n+$ etc. is written $d_x^l\, d_y^m\, d_z^n$ etc. u. This notation which Hind describes is that of Cauchy for partial *differentials*.

M. Ohm. The much praised and much criticized Martin Ohm,† in Germany, advanced notations as follows:

If f is a function of $x, y, z, \ldots$, then the partial derivatives with respect to x are indicated by $\partial f_x, \partial^2 f_x, \partial^3 f_x, \ldots \partial^a f_x$. Similarly for partial derivatives with respect to y or z, etc. More generally, $\partial^{a,b,c} f_{x,y,z}$ means the ath partial derivative with respect to x, followed by the bth partial derivative with respect to y, and then by the cth partial derivative with respect to z. Ohm writes the form $\partial^{o,a,b} f_{x,y,z}$, and also $\partial^{a,b} f_{y,z}$; the latter notation is shorter but does not display to the eye which of the variables is constant. Ohm's is perhaps the fullest notation for partial differentiation developed up to that time. But Ohm was not content to consider simply functions which are expressed directly and explicitly in terms of their variables; he studied also functions $f_{x,y,z,\cdots}$ in which some of the variables $x, y, z, \ldots$ appear implicitly, as well as explicitly. For instance, p. 129, let $f = 3x^2 + 4xy - 5y^2 z^2 + 4xyz$, where $y = a + bx$ and $z = px^3$. He introduces the notation $f_x = 3x^2 + 4xy - 5y^2 z^2 + 4xyz$, where f_x is considered as the explicit function of x, and y and z are to be taken constant or independent of x; he represents by $f_{(x)}$ the above expression when account is taken of $y = a + bx$ and $z = px^3$. Hence $\partial f_x = 6x + 4y + 4yz$, while $\partial f_{(x)} = 4a + (8b + 6)x - 30a^2 p^2 x^5 - 70abp^2 x^6 - 40b^2 p^2 x^7 + 16apx^3 + 20bx^4$. The two partial derivatives are wholly different; the former is an incomplete partial derivative, the latter is a complete partial derivative. He states (p. 276) that in 1825 he brought out in Berlin his "Lehre vom Grössten und Kleinsten", and introduced there a special notation for the complete and incomplete partial differentials which correspond to the complete and incomplete partial derivatives, $f_{(x)}$ and f_x. In 1825 he marked the incomplete partial differentials $\frac{df}{dv} \cdot dv$; the complete partial differentials $\frac{df}{dv} \cdot dv$. Similarly for higher orders, Ohm states in 1829 (p. 276) that he has found his notation $\frac{df}{dx}$ adopted by other writers.

* John Hind, Principles of the Differential Calculus, 2d Ed., Cambridge, 1831, p. 372.
† Martin Ohm, System der Mathematik, vol. III, 1829, p. 118, 119.

In the 1829 publication Ohm considers a great many cases of this type and gives a diffuse presentation of partial differentiation covering over 160 pages. He compares his notations with those of L. Euler and A. Fontaine. $\frac{df}{dx}$ meant with Euler a complete derivative, with Fontaine a partial derivative. As previously seen, Fontaine once marked a complete derivative $\frac{1}{dx} \cdot d\mu$, but generally he dealt with complete *differentials* rather than complete *derivatives*.

If f is a function of x and y, then the three notations are exhibited by Ohm as follows (p. 183):

$$df = df_x + df_y \qquad \text{according to M. Ohm,}$$

$$df = \left(\frac{df}{dx}\right) \cdot dx + \left(\frac{df}{dy}\right) \cdot dy \qquad \text{according to L. Euler,}$$

$$df = \frac{df}{dx} \cdot dx + \frac{df}{dy} \cdot dy \qquad \text{according to A. Fontaine,}$$

where df means the complete differential. Ohm warns the reader (p. 182) that in Fontaine's notation for differentials the dx's and dy's cannot be cancelled. We see in Ohm's intellectual ménage a large progeny of symbols, but it was a group with no one to bid them welcome.

W. R. Hamilton. About the same time that M. Ohm was devising suitable symbols in Germany, William Rowan Hamilton in Great Britain felt the need of a suitable symbolism. At first he used parentheses* as Euler had done; they are found in an article which was written in 1824 and printed in 1828. In the table of contents of this article, probably prepared when it was printed, the partial derivatives appear in the garb $\frac{\delta i}{\delta x}$. This delta notation is used by Hamilton† in an article printed in 1830, and another printed in 1834,‡ where the expressions are called "partial differential coefficients." On account of the close resemblance between δ and ∂, this notation is almost identical with that proposed by Legendre forty years earlier.

W. Bolyai. Quaint notations are found in Wolfgang Bolyai's§ *Tentamen* 1832. He designates by $\mathfrak{D}x$ a function of x, by ${}^{n}_{\circ}\mathfrak{D}x$ the nth differential, by ${}^{n}_{\circ}\mathfrak{D}x, y$ or $\circ^{n}_{x}\mathfrak{D}x, y$ the nth partial differential with respect to x.

* W. R. Hamilton in Transactions of the Royal Irish Academy, vol. XV, Dublin, 1828, p. 152.
† Loc. cit., vol. XVI, 1830, p. 2.
‡ Philosophical Transactions of Royal Society, London, 1834, p. 249.
§ Wolfgangi Bolyai de Bolya, Tentamen, 2. Ed., 1897, p. 207, 640.

Cauchy and Moigno. Soon after this time, A. Cauchy in Paris slightly modified his former symbols. In 1840 L'Abbé Moigno* published a book on the calculus "d'après les méthodes et les ouvrages publiés ou inédits de M. A.-L. Cauchy." For higher partial differentials and derivatives he gives the notations $d_x^2 d_y d_z u$, $d_{x^2yz}^4$, $\frac{d^4 u}{dx^2\, dy\, dz}$. That these notations do not always distinguish the partial derivative from the total is evident from what is given on p. 444 as partial derivatives: $\frac{dz}{dx}$, $\frac{dz}{dy}$, $\frac{d^2 z}{dx^2}$, etc. That Moigno was not satisfied with his notations is evident from our previous quotations from his Introduction.

In the same year (1840) Cauchy himself brought out in Paris his Exercices d'analyse et de physique mathématique, where (p. 5) he lets D_x, D_y, D_z, D_t stand for the partial derivatives of a function of the independent variables x, y, z, t; D_t^2 stands for the second partial derivative.

More fully developed is his notation† of 1844. Taking s as a function of many variables, x, y, z, ... u, v, w, Cauchy designates

partial increments by	$\Delta_1 s$, $\Delta_{11} s$, $\Delta_{111} s$, ...
and "différentielles partielles" by	$d_1 s$, $d_{11} s$, $d_{111} s$, ...
or, if of the second order, by	$d_1 d_{11} s$, $d_1 d_{111} s$, ...
When the variables are specified, by	$\Delta_x s$, $\Delta_y s$, $\Delta_z s$, ...
"Différentielles partielles" by	$d_x s$, $d_y s$, $d_z s$, ...
When of the second order, by	$d_x d_y s$, $d_x d_z s$, ...
"Dérivées partielles" by	$D_x D_y s$, $D_x D_z s$, ...

and so on.

The $D_x s$ as a partial derivative is distinguished from a total derivative by the x, provided that s is a function of more than one variable; otherwise the $D_x s$ and Ds mean the same. One objection to this symbolism is that the variables considered as constant during the differentiation are not exhibited to the eye.

C. G. J. Jacobi. We come now to the researches of C. G. J. Jacobi who is usually credited with the invention of the notation of partial derivatives which has become popular in recent years. Had he instituted a careful historical survey of the notations which had been proposed by mathematicians before him, he would have found much of value — notations of Euler, Fontaine, Karsten, Legendre, Lagrange of 1797 and of 1801, Da Cunha, Arbogast, Crelle, Barlow, Ohm, Hamilton and Cauchy. But Jacobi made no such survey. He happened to know the notations of Euler and of Lagrange of 1797. Not

* L'Abbé Moigno, Leçons de Calcul différentiel et de calcul intégral, Tome I, Paris, 1840, p. 118–121.

† A. Cauchy, Exercices d'analyse et de physique mathématique, Paris, 1844, p. 12–17.

satisfied with these he devised one of his own. As is so common in the history of science, his invention happened to have been made by others before him. Nevertheless, the mathematical public, uncritical on matters of priority, is crediting Jacobi with devices first proposed by Legendre and Hamilton.

It was in 1841 that Jacobi published his paper* *De determinantibus functionalibus*, in which he introduces d and ∂ for total and partial derivatives (differentialia partialia), respectively. He says: "To distinguish the partial derivatives from the ordinary, where all variable quantities are regarded as functions of a single one, it has been the custom of Euler and others to enclose the partial derivatives within parentheses. As an accumulation of parentheses for reading and writing is rather onerous, I have preferred to designate ordinary differentials by the character d and the partial differential by ∂. Adopting this convention, error is excluded. If we have a function f of x and y, I shall write accordingly,

$$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy,$$

... This distinction may be used also in the designation of integrations, so that the expressions

$$\int f(x, y)\,dx, \qquad \int f(x, y)\,\partial x$$

have different meaning; in the former y and therefore also $f(x, y)$ are considered functions of x, in the latter one carries out the integration according to x only and one regards y during the integration as a constant."

Jacobi proceeds to the consideration of a function of many variables. "Let f be a function of $x_0, x_1, \ldots, x_n$. Assume any n functions $w_1, w_2, \ldots, w_n$ of these variables and think of f as a function of the variables $x_1, w_1, w_2, \ldots, w_n$. Then, if $x_1, x_2, \ldots, x_n$ remain constant, the $w_1, w_2, \ldots, w_n$ are no longer constants when x varies, and also, when $w_1, w_2, \ldots, w_n$ remain constant, $x_1, x_2, \ldots, x_n$ do not remain constant. The expression $\frac{\partial f}{\partial x}$ will designate entirely different values according as these or other magnitudes are constant during the differentiation." To simplify matters, Jacobi stipulates: "If I say that f is a function of the variables $x_1, x_2, \ldots, x_n$, I wish it to be understood that, if this function is differentiated partially, the differentiation shall be so carried out that always only one of these varies, while all the others remain constant."

* Journal für die reine und angewandte Mathematik, vol. 22, p. 319; also vol. 23, p. 4. See *Ostwald's Klassiker* No. 78.

Jacobi proceeds: "If the formulas are to be free from every ambiguity, then also there should be indicated by the notation, not only according to which variable the differentiation takes place, but also the entire system of independent variables whose function is to be differentiated partially, in order that by the notation itself one may recognize also the magnitudes which remain constant during the differentiation. This is all the more necessary, as it is not possible to avoid the occurrence in the same computation and even in one and the same formula, of partial differentials which relate to different systems of independent variables, as for example, in the expression given above, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x}$ in which f is taken as a function of x and u, u on the other hand is a function of x and y. The $\frac{\partial f}{\partial x}$ (derived from $f(x, y)$) changed into this expression when u was introduced in the place of y as an independent variable. If however we write down, besides the dependent variable, also the independent variables occurring in the partial differentiations, then the above expression can be represented by the following formula which is free of every ambiguity:

$$\frac{\partial f(x, y)}{\partial x} = \frac{\partial f(x, u)}{\partial x} + \frac{\partial f(x, u)}{\partial u} \cdot \frac{\partial u(x, y)}{\partial x}.$$

By way of criticism of this notation, P. Stäckel* adds in his notes on Jacobi's paper: "This notation is ambiguous too, for the symbol is used in two different meanings, in as much as $f(x, y)$ is another function of x and y than is $f(x, u)$ of x and u." Jacobi proceeds to remark that one can conceive of complicated relations in which it would be exceedingly onerous to completely define partial derivatives by the notation alone; a formula written in only one line might have to be expanded so as to fill a page. To avoid such differences it is best in complicated cases to dispense with the designation in the formula of all the independent variables. To distinguish two systems of partial derivatives which belong to different systems of variables, one system of derivatives may be enclosed in parenthesis, as was the practice with Euler.

The notation $\frac{\partial u}{\partial x}$ advocated by Jacobi did not meet with immediate adoption. It took half a century for it to secure a generally recognized place in mathematical writing. In 1844 Jacobi used it in vol. 29 of Crelle's Journal. When in 1857 Arthur Cayley† abstracted Jacobi's paper, Cayley paid no heed to the new notation and wrote all derivatives in the form $\frac{du}{dx}$.

* Ostwald's Klassiker, No. 78, p. 65.

† A. Cayley, "Report on recent Progress in Theoretical Dynamics" in British Association Report, 1857.

G. A. Osborne refers in the preface of his *Differential and Integral Calculus*, Boston, 1895, to this notation as having "recently come into such general use."

O. Hesse. A few years after Jacobi had published the article containing the $\frac{\partial z}{\partial x}$ notation, Otto Hesse advanced another notation which has been found useful in analytical geometry. In his paper on points of inflexion of cubic curves* he designates $\frac{\partial^2 u}{\partial x_\varkappa \partial x_\lambda}$ of the function u for brevity by $u_{\varkappa,\lambda}$. In accordance with this plan G. Salmon† writes the first partial derivatives of U with respect to x, y, and z, respectively, by U_1, U_2, U_3 and the second partial derivatives by U_{11}, U_{22}, U_{33}, U_{23}, U_{31}, U_{12}; or as it is expressed in Fiedler's edition, U_i is the first partial derivative of U with respect to x_i.

B. Peirce. Arbogast's and Cauchy's symbolism was in part adopted in the United States by Benjamin Peirce‡ of Harvard, who introduced it thus: "Denoting by $D_{\cdot x}\cdot$, $D_{\cdot u}\cdot$, the differential coefficients taken on the supposition that x, u are respectively the independent variables, we have, at once, $D_{\cdot x}y = \frac{dy}{dx} = \frac{D_{\cdot u}y}{D_{\cdot u}x}$." Accordingly, D signifies the total derivative, $D_x y$ the partial derivative with respect to x.

From an anonymous article in the Mathematical Monthly§ (probably from the pen of its editor, J. D. Runkle) we cull the following comments:

"The student will observe, that $\frac{du}{dx}$ denotes the derivative of the quantity u; but the symbol, as separated from the quantity, and simply denoting the operation, is $\frac{d}{dx}$. Thus, $\frac{d}{dx} f(x)$ tells us to find the derivative of $f(x)$. The inconvenience of the use of the symbol $\frac{d}{dx}$, in this and like cases has led to the adoption of D in its place. If we wish to indicate at the same time the particular variable, as x, in reference to which the derivative is to be taken, then the symbol D_x is used. ... In the case of the general function, as $f(x)$, the notation $f'(x)$, $f''(x)$, $f'''(x)$, etc. to denote the successive derivatives, was used by Lagrange, and is most convenient. So far as we know, Prof. Peirce is the only author in this country who has used D; and we have made these remarks for the benefit of those students who meet with this notation only in the Monthly. ... Dx^n is better than $\frac{d}{dx} x^n$."

* Otto Hesse in Crelle's Journal, 1843, vol. 28, p. 99; Gesammelte Werke, München, 1897, p. 124.

† George Salmon, Higher Plane Curves, 3d Ed., Dublin, 1879, p. 49, 50.

‡ B. Peirce, Curves, Functions and Forces, vol. I, 2. Edition, Boston, 1852, p. 225.

§ Mathematical Monthly, Edited by J. D. Runkle, vol. I, Cambridge, 1859, p. 328.

G. W. Strauch. A special symbolism to distinguish between two partial derivatives with respect to x, one recognizing only explicit forms of x, the other observing also the implicit forms, was given by the Swiss, G. W. Strauch.* He assumes $U = \varphi(x, y, z)$ and $z = \psi(x, y)$. Following in part Cauchy's notation, Strauch uses the symbol $\frac{d_x U}{dx}$ to represent the "incomplete" partial differential coefficient with respect to x which is obtained by differentiating U only with respect to the explicit x; he uses the symbol $\frac{\overline{d_x U}}{dx}$ to represent the "complete" partial differential coefficient with respect to x which is obtained by differentiating U with respect to the implicit as well as the explicit x. He has the following relation between the partial derivatives,

$$\frac{\overline{d_x U}}{dx} = \frac{d_x U}{dx} + \frac{d_z U}{dz} \cdot \frac{d_x z}{dx}.$$

It will be remembered that the double horizontal line used here had been employed for the same purpose in 1825 by Martin Ohm.

J. M. C. Duhamel. The D_x, D_y, etc. suggested by Cauchy for partial differentiation was used in many quarters. For example, Ernest Pfannenstiel,† in 1882, wrote $D_x^2 z$, $D_{x,y}^2 z$, etc., to represent partial derivatives in a paper on partial differential equations.

Duhamel‡ in ordinary cases used no distinctive device for partial differentiation, but when many variables and derivatives of any orders were involved, he designated "par $F_{x,y,x\ldots}^{m,n+p\ldots}(x, y)$ ou par $D_{x,y,x\ldots}^{m+n+p\ldots} u$ le résultat de m dérivations partielles effectuées par rapport à x sur la fonction $u = F(x, y)$, suivies de n dérivations partielles du résultat par rapport à y, lesquelles seront elles-mêmes suivies de p dérivations par rapport à x; et ainsi de suite." For partial differentials he wrote similarly, $d_{x,y,x\ldots}^{m+n+p\ldots} u$. It will be observed that, strictly speaking, there is nothing in this notation to distinguish sharply between partial und total operations; cases are conceivable in which each of the $m + n + p \cdots$ differentiations might be total.

Ch. Méray. An obvious extension of the notation used by Duhamel was employed by Ch. Méray.§ Representing by $p, q, \ldots$ the orders of the partial derivatives with respect to the independent variables $x, y, \ldots$, respectively,

* G. W. Strauch, Praktische Anwendungen für die Integration der totalen und partialen Differentialgleichungen, Braunschweig, 1865, p. 7—13.

† Societät der Wissenschaften zu Upsala, Sept. 2, 1882.

‡ J. M. C. Duhamel, Éléments de calcul Infinitésimal, Tome I, 3. Édition, Paris, 1874, p. 267.

§ Ch. Méray, Leçons nouvelles sur l'analyse infinitésimale, vol. I, Paris, 1894, p. 123.

and by the sum $p+q+\ldots$ the total order of the derivatives, he suggests two forms, either

$$f^{(p,q,\ldots)}_{x,y,\ldots}(x,y,\ldots) \quad \text{or} \quad D^{(p,q,\ldots)}_{x,y,\ldots} f(x,y,\ldots).$$

Applying this notation to partial differentials, Méray writes,

$$d^{(p,q,\ldots)}_{x,y,\ldots} f = f^{(p,q,\ldots)}_{x,y,\ldots}(x,y,\ldots)\, dx^p\, dy^q \ldots$$

Méray's notation is neither elegant, nor typographically desirable. It has failed of general adoption. In the *Encyclopédie des sciences mathématiques** J. Molk prefers an extension of the Legendre-Jacobi notation; thus $\frac{\partial^k u}{\partial x^p \partial y^q \ldots}$, where $k = p+q+\cdots$.

T. Muir. Thomas Muir stated† in 1901 that he found it very useful in lectures given in 1869 to write $\Phi\overline{x,y,z}$, in place of $\Phi(x,y,z)$ and to indicate the number of times the function had to be differentiated with respect to any one of the variables by writing that number above the vinculum; thus, $\Phi\overline{\begin{matrix}1&3&2\\x,&y,&z\end{matrix}}$ meant the result of differentiating once with respect to x, thrice with respect to y, and twice with respect to z. Applying this to Jacobi's example, in which $z = \Phi\overline{x,y}$, we should have satisfactorily $\frac{\partial z}{\partial x} = \Phi\overline{\begin{matrix}1\\x,&y\end{matrix}}$; but when there is given $z = \Phi\overline{x,u}$ and $u = \Psi\overline{x,y}$, then arises the serious situation that we are not certain of the meaning of $\frac{\partial z}{\partial x}$, as it would stand for $\Phi\overline{\begin{matrix}1\\x,&u\end{matrix}}$ or for $\Phi\overline{\begin{matrix}1\\x,&u\end{matrix}} + \Phi\overline{\begin{matrix}1\\x,&u\end{matrix}} \cdot \Psi\overline{\begin{matrix}1\\x,&y\end{matrix}}$, according as u or y was to be considered constant. In such cases he declares the notation $\frac{\partial z}{\partial x}$ inadequate.

Thomas Muir explained his notation also in *Nature*, vol. 66, 1902, p. 271, but was criticized by John Perry who, taking a unit quantity of mere fluid, points out that v, p, t, E, Φ are all known if any two (except in certain cases) are known, so that any one may be expressed as a function of any other two. Perry prefers the notation $\left(\frac{dE}{dv}\right)_p$, which is the Eulerian symbol to which p is affixed; according to Muir's suggestion, Perry says, one must let $E = f\overline{v,p}$ and write $f\overline{\begin{matrix}1\\v,&p\end{matrix}}$. To this remark, Muir replies (p. 520) that he would write

* Tome II, vol. I, p. 296.

† Thomas Muir in Proceedings of the Royal Society of Edinburgh, vol. 24, 1901–1903, p. 162

$E\frac{1}{r,p}$, or, if a vinculum seems "curious," he would write $E(\frac{1}{r}, p)$. To these proposals A. B. Basset remarks (p. 577): "Dr. Muir's symbols may be very suitable for manuscripts or the blackboard, but the expense of printing them would be prohibitive. No book in which such symbols were used to any extent could possibly pay. On the other hand, the symbol $(dE/dv)_p$ can always be introduced into a paragraph of letter-press without using a justification or a vinculum; and this very much lessens the expense of printing."

P. Mansion. Muir and Perry were not alone in their rejection of the notation used by Jacobi in 1841. Paul Mansion,* for instance, in 1873 prefers to use the straight letter d for partial derivatives; thus, $\frac{dz}{dx_1}$, $\frac{dz}{dx_2}$, where z is an explicit or implicit function of the independent variables $x_1, x_2, \ldots$. He uses also $\frac{\delta q}{\delta x_1}$, $\frac{\delta q}{\delta x_2}$ to designate "the derivatives of an *explicit* function $q\,(x_1, x_2, \ldots)$ of $x_1, x_2, \ldots$ with respect to the letter X_1, to the letter X_2, etc., without considering whether $x_1, x_2, \ldots$ are independent of each other or not."

3. Symbols for Fluents and Integrals.

G. W. Leibniz. In a manuscript† dated October 29, 1675, Leibniz introduces the symbol $\int l$ for *omn. l*, that is, for the sum of the l's. As a facsimile reproduction was published by C. I. Gerhardt in 1899, we have definite information as to the exact form of the $\int$. Not until eleven years later, did Leibniz use the symbol in print. In 1686 Leibniz used it eight times in an article in the Acta eruditorum, p. 297, 299, but in that paper it did not quite take its modern form; the lower part was amputated and the sign appeared thus, ∫. It was simply the small letter "*s*", as printed at that time. It resembled the modern type form for "f". He writes the cycloidal formula thus: $y = \text{l}\,\overline{2x - xx} + \int dx$; $\text{l}\,\overline{2x - xx}$. This form of the symbol appears in the Acta eruditorum for 1701, p. 280, where Louis Carré of Paris writes $a \int dx = ax$, and again in 1704, p. 313, in a reprint of John Craig's article from the Philosophical Transactions of London, No. 284, viz. $\int : z\,dy$. We have noticed this form also in works of G. Manfredi‡ at Bologna in 1707 and of Chr. Wolf§ at Halle in 1713.

* Théorie des équations aux dérivées partielles du premier ordre par M. Paul Mansion, professeur a l'université de Gand, 1873, Tome XXV.

† C. I. Gerhardt, Entdeckung der höheren Analysis, Halle, 1855, p. 125.

‡ De Constructione aequationum differentialium primi gradus. Authore Gabriele Manfredio, Bononiae, 1707, p. 127.

§ Elementa matheseos universae, tomus I ... Autore Christiano Wolfo. Halae, 1713, p. 474.

In Leibnizens Mathematische Schriften, edited by C. I. Gerhardt, vol. 5, 1858, p. 231, the notation of the article of 1686 is not reproduced with precision; the sign of integration assumes in the reprint the regular form $\int$, namely a slender, elongated form of the letter. John Bernoulli,* who was the first to use the term "integral," proposed in his correspondence with Leibniz the letter I as the symbol of integration, but finally adopted $\int$ in deference to Leibniz. We cannot agree with Klügel,† that I would have been more appropriate (schicklicher); this shorter symbol possessed less adaptability to the nineteenth century need of indicating symbolically the limits of integration. In articles published in the Acta Eruditorum of 1694 and 1695, Leibniz places a comma after the $\int$ thus $\int, xx\,dx$, but Johann Bernoulli omits the comma in the volume for 1698, p. 53. Some eighteenth and early nineteenth century writers placed a dot after the sign, thus $\int . .$ or the colon, as in $\int : :$ E. Waring‡ wrote $\int . \frac{\dot{x}}{x} (\log. x)$. In 1755 Euler§ wrote $\int p\,dx \int q\,dx \int r\,dx \ldots$, which did not indicate a product like $\int p\,dx \cdot \int q\,dx \cdot \int r\,dx \ldots$ but was to be interpreted so that $\int$ applies to all that follows — a relation which some authors (Gherli for instance, as we shall see) carefully indicate by the use of vinculums.

As regards the differential in the expression to be integrated, the usual practice has been to write it down, thus $\int y\,dx$, but this practice has not been universal. Leibniz's conception of an integral as a summation would seem to require the use of the differential. In his printed paper of 1686 he adopts the form $\int p\,dy$, although in his manuscript of October 29, 1675 he had written $\int x^2 = \frac{x^3}{3}$. If integration is conceived as the converse of differentiation, then no serious objection can be raised to the omission of the differential. And so we find a few authors who prefer to omit the dx, as for instance, Benjamin Peirce,‖ who writes $(n+1)\int . ax^n = ax^{n+1}$. On Oct. 11–16, 1858,

* Johann Bernoulli to Leibniz, April 7, 1696, Leibnizens Mathematische Schriften, Ed. C. I. Gerhardt, vol. 3, Halle, 1855, p. 262, 273.

† G. S. Klügel, Mathematisches Wörterbuch, 1. Abth., 2. Theil, Leipzig, 1805, Art. "Integral," p. 746.

‡ Edward Waring, Meditationes analyticae, 1786, p. 23.

§ L. Euler, Institutiones Calculi Integralis, IV, Suppl. IX, Art. 37.

‖ B. Peirce, Curves, Functions, and Forces, vol. II, Boston, 1846, p. 69.

William R. Hamilton wrote to P. G. Tait,* "And perhaps you may have adopted, even publicly — as Airy has done, using the (to me) uncouth notation $\int_{\theta}(\quad)$ for $\int(\quad)d\theta$ — the system which rejects differentials."

I. Newton. In his Quadratura curvarum,† of 1704, Newton gave the symbol $\dot{x}$ as the integral of x, x as the integral of $\dot{x}$. In fact, in the succession, $\dot{x}$, $\dot{\dot{x}}$, x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, each term was the fluxion (fluent) of the preceding (succeeding) term. Another notation for integral was the enclosure of the term in a rectangle‡, as is explained in Newton's De Analyse per numero terminorum infinitas, where $\boxed{\frac{aa}{64x}}$ stands for $\int \frac{aa \cdot dx}{64x}$.

That Newton's notation for integration was defective, is readily seen. The x was in danger of being mistaken for an abscissa in a series of abscissas x, x', x''; the rectangle was inconvenient in preparing a manuscript and well-nigh impossible for printing, when of frequent occurrence. As a consequence, Newton's signs of integration were never popular, not even in England. They were used by Brook Taylor, in his Methodus Incrementorum, London, 1715; he writes (p. 2) "$\dot{x}$ designat fluentem secondam ipsuis x," and (p. 38), "$\dot{p} = -r\dot{s}$, adeoque $p = -\boxed{r\dot{s}}$."

In Newton's Principia, 1687 (Book II, Lemma II) fluents are represented simply by capital letters and their fluxions by the corresponding small letters. Newton says: "If the moments of any quantities A, B, C, etc. increasing or decreasing, by a perpetual flux, or the velocities of the mutations which are proportional to them, be called a, b, c, etc., the moment or mutation of the generated rectangle AB will be $aB + bA$." Here a velocity or fluxion is indicated by the same symbol as a moment. With Newton a "fluxion" was always a *velocity*, not an infinitely small quantity, a "moment" was usually, if not always, an infinitely small quantity. Evidently, this notation was intended only as provisional. Maclaurin does not use any regular sign of integration. He says§ simply: "$\dot{y}z + \dot{z}y$, the fluent of which is yz." Nor have we been able to find any symbol of integration in Thomas Simpson's Treatise of Fluxions, London, 1737, 1750, in Edmund Stone's Integral Calculus,‖ in William Hale's Analysis Fluxionum, 1804, in John Rowe's Doctrine of Fluxions, 4th Ed., London, 1809, S. Vince's Principles of Fluxions, Philad., 1812.

* C. G. Knott's Life and Scientific Work of Peter Guthrie Tait, Cambridge, 1911, p. 121. For this reference and also some other data in this article I am indebted to Professor J. H. M. Wedderburn.

† I. Newton, Quadratura curvarum, London, 1704. Opera, vol. 1, p. 338.

‡ I. Newton, Opera, Horsley's Ed., vol. I, 1779, p. 272.

§ C. Maclaurin, Treatise of Fluxions, Bk. II, 1742, p. 600.

‖ We have examined the French translation by Rondet, under the title Analyse des infiniment petits comprenant le Calcul Intégral, par M. Stone, Paris, 1735.

Other English authors employed the Continental symbol $\int$. Even such a bitter opponent of Leibniz as John Keill* used the $\int$, while for differentiation he adhered to the Newtonian $\dot{x}$, $\ddot{x}$, etc. This mixture of British and Continental symbolism is found regularly in the writings of Edward Waring,† and much later in those of Olinthus Gregory,‡ who wrote $\int x y \dot{x}$, and those of John Brinkley, astronomer in Dublin.§ Brinkley uses the $\int$ with a horizontal bar added, as in $f \dot{\varrho}\, s$.

Ch. Reyneau and others. Perhaps no mathematical symbol has encountered so little competition with other symbols, as has $\int$: the sign $\int\!\!\!\!\!\int$ can hardly be called a competitor, it being simply another form of the same letter. The $\int\!\!\!\!\!\int$ was used in France by Charles Reyneau‖ in 1708 and by L'Abbé Sauri in 1774; in Italy by Paolo Frisi,¶ by Odoardo Gherli** who, in the case of multiple integrals, takes pains to indicate by vinculums the scope of each integration, thus $\int\!\!\!\!\!\int \cdot dx \int \cdot dx \int \cdot y\, dx$. In R. G. Boscovich's†† treatise of 1796, there is used part the time a small s, as in $s\, a x^m dx = \frac{a x^{m+1}}{m+1}$, then $\int y dx$, and finally $\int\!\!\!\!\!\int \frac{cy^2 dx}{2r}$, this large letter being used, apparently, because there was plenty of space in front of the fraction. In the books which we have quoted the use of $\int\!\!\!\!\!\int$ rather than $\int$ was probably due to the greater plenty of the former type in the respective printing offices.

Not infrequently the two symbols were used in the same publication, the $\int\!\!\!\!\!\int$ to indicate some specialized integral. Thus, Joseph Fourier said in an

* J. Keill in Philosophical Transactions for 1714–16, paper dated Nov. 24, 1713.

† E. Waring, Meditationes analyticae, 1785, p. 22.

‡ Olinthus Gregory, Treatise on Mechanics, vol. I, 3. Ed., London, 1815, p. 62.

§ Transactions of Royal Irish Academy, vol. 12, 1815, p. 85. See also vol. 13, 1818, p. 31.

‖ Ch. Reyneau, Usage de L'Analyse, Tome II, Paris, 1708, p. 734.

¶ Paulli Frisii Operum Tomus primus, Milan, 1782, p. 303.

** O. Gherli, Gli elementi teorico-pratici delle matematiche pure, Tomo VI, Modena, 1775, p. 1, 334.

†† Ruggero Guiseppe Boscovich, Elementi delle matematiche pure Edizione terza Italiana, Venezia, 1796, p. 477, 479, 484.

article of 1811, first published in 1824:* "and taking the integral from $x = 0$ to $x = \pi$, one has, on representing these integrations by the sign $\int$, $\int q \sin . i x \, dx = \ldots$." Further on he writes "$a_0 = \frac{1}{2} \int x \, dx$, ou $\frac{\pi^2}{4}$." It will be seen that later Fourier suggested the notation for definite integrals now in general use. L. M. H. Navier† in an article on fluid motion remarks: "Le signe $\int$ désigne une intégration effectuée, dans toute l'entendu de la surface du fluide ...." In the third edition of Lagrange's Mécanique Analytique‡ we read: "Nous dénoterons ces intégrales totales, c'est-à-dire relatives à l'étendue de toute la masse, par la charactéristique majuscule $\int$, en conservant la charactéristique ordinaire $\int$ pour désigner les intégrales partielles ou indéfinies." Finally, we refer to C. Jordan, who, in his Cours d'Analyse, I, Paris, 1893, p. 37 uses $\int_E$ to mark "l'intégrale de la fonction f dans le domaine E," and says that it is generally designated thus.

A. L. Crelle. We have encountered only one writer on the European continent who deliberately rejected Leibniz's sign $\int$ or $\int$, namely A. L. Crelle§ of Cassel, the founder of "Crelle's Journal." In 1813 he rejected the symbol as foreign to the nature of the subject. He argued that since in differentiation (or the finding of dy, d^2y) d appears in the position of a multiplier, the symbol for the inverse operation, namely, integration, should be a d placed in the position of a divisor, thus $\frac{1}{d}$, $\frac{1}{d^2}$, ... Like Leibniz, Crelle was influenced in his selection by his conception of the operation called integration. He looked upon it as the *inverse* of differentiation, and chose his symbol accordingly; Leibniz saw this inverse relation; nevertheless he looked upon integration primarily as a summation and accordingly chose the $\int$ as the first letter in *summa*. Crelle found no following whatever in his use of $\frac{1}{d}$.

L. Euler. Limits of integration were at first indicated only in words. Euler

* Mémoires de l'académie r. des sciences de L'Institut de France, Tome IV, années 1819 et 1820, Paris 1824, p. 303, 309.

† Mémoires de l'académie r. des sciences de L'Institut de France, année 1823, Tome VI, Paris 1827, p. 412.

‡ J. L. Lagrange, Œuvres, vol. 11, p. 85.

§ A. L. Crelle, Darstellung der Rechnung mit veränderlichen Grössen, Erster Band, Göttingen, 1813, p. 88, 89.

was the first to use a symbol in his integral calculus,* of which the following is an illustration:

$$Q = \int_0^{\cdot} \frac{x^{p-1}\,\partial x}{\mathbf{1}^{\cdot}(1-x^n)^{n-q}} \qquad \begin{bmatrix} ab\,x^n = \frac{1}{2} \\ ad\,x = 1 \end{bmatrix}.$$

This notation, with the omission of the "ab" and "ad," was used in 1819 or 1820 by J. Sarrus† of Strasbourg and H. G. Schmidten.‡

J. Fourier. Our modern notation for definite integrals constituted an important enrichment of the notation for integration. It was introduced by Joseph Fourier who was an early representative of that galaxy of French mathematical physicists of the early part of the 19th century. In Fourier's epochmaking work of 1822 on La Theorie analytique de la chaleur§ he says: "Nous désignons en général par le signe $\int_a^b$ intégrale qui commence lorsque la variable équivalent à a, et qui est complète lorsque la variable équivant à b; et nous écrivons l'équation (u) sous la forme suivante

$$\frac{\pi}{2}\,\varphi(x) = \frac{1}{2}\int_0^{\cdot\pi} \varphi(x)\,dx + \text{ etc.}"$$

But Fourier had used this notation somewhat earlier in the *Mémoires* of the French Academy for 1819—1820, in an article of which the early part of his book of 1822 is a reprint. The notation was adopted immediately by G. A. A. Plana‖ of Turin, who writes $\int_0^1 a^u\,du = \frac{a-1}{Log.\,a}$, by A. Fresnel¶ in one of his memorable papers on the undulatory theory of light, and by Cauchy.** This instantaneous display to the eye of the limits of integration was declared by S. D. Poisson†† to be a "notation très-commode."

Ohm (op. cit., vol. 4, 1830, p. 137) prefers $\int_{\beta-\alpha} \varphi\,.\,dx$ to Fourier's $\int_\alpha^\beta \varphi\,.\,dx$.

It was F. Sarrus‡‡ who first used the signs $F(x)\,{}_a^x$ or ${}_a^x F(x)$ to indicate the

* L. Euler, Institutiones calculi integralis, 3rd. Edition, vol. IV, 1845, Supplementum V, p. 324. [First edition, 1768–1770].

† Gergonne's Annales de mathématique, vol. 12, 1819 and 1820, p. 36.

‡ Loc. cit., p. 211.

§ Œuvres de Fourier (Edited by G. Darboux), Tome 1, Paris, 1888, p. 231. See also p. 216.

‖ J. D. Gergonne, Annales de mathématiques, vol. 12, Nismes, 1819 and 1820, p. 148.

¶ Mémoires de l'Académie r. des sciences de l'Institut de France, Tome V, années 1821 et 1822, Paris, 1826, p. 434.

** Cauchy, Résumé des leçons données a l'école royale polytechnique sur la calcul infinitésimal, Paris, 1823, or Œuvres, 2. Série, vol. 4, p. 188.

†† Mémoires de l'académie des sciences de L'Institut de France, Tome VI, année 1823, Paris, 1827, p. 574.

‡‡ F. Sarrus in Gergonne's Annales, vol. 14, 1823, p. 197.

process of substituting the limits a and x in the general integral $F(x)$. This designation was used later by Moigno and Cauchy.

In Germany Martin Ohm advanced another notation for definite integrals, viz., $\int_{x \div a} \varphi \, . \, dx$ or $(\partial^{-1}\varphi_x)_{x \div a}$, where x is the upper limit and a the lower. He adhered to this notation in 1830 and 1846, claiming for it greater convenience over that of Fourier in complicated expressions.*

Slight modification of the notation for definite integration was found desirable by workers in the theory of functions of a complex variable. For example, A. R. Forsyth† writes $\int_B$ where the integration is taken round the whole boundary B.

V. Volterra and G. Peano. What C. Jordan‡ calls "l'intégrale par excès et par défaut" is represented by Vito Volterra§ thus $\int_{x_0}^{x_1}$ and $\int_{x_0}^{x_1}$. G. Peano,‖ who uses $\int$ as the symbol for integration, designates by $\int (f, a|\!-\!|\, b)$ the integral of f, extended over the interval from a to b, and by $\int^1 (f, a|\!-\!|\, b)$ the "intégrale par excès," and by $\int_1 (f, a|\!-\!|\, b)$ the "intégrale par défaut."

4. Symbols for Passing to the Limit.

S. Lhuilier. The abbreviation "lim." for limit does not seem to have been used before Simone Lhuilier¶ who writes "lim. $q : Q$" and "Lim. $\frac{\Delta y}{\Delta x}$." It was used by Garçaõ Stockler of Lisbon in 1794, by L. N. M. Carnot** in 1797 (who used also L), by John Brinkley†† of Dublin in 1817, and by A. L. Cauchy‡‡ in 1821. Cauchy points out that "lim. $(\sin . x)$" has a unique value 0, while "lim. $\left(\left(\frac{1}{x}\right)\right)$" admits of two values, and "lim. $\left(\left(\sin . \frac{1}{x}\right)\right)$" of an infinity of values, the double parentheses being used to designate all the values that the enclosed function may take, as x approaches zero.

* Martin Ohm, Lehre vom Grössten und Kleinsten, Berlin, 1825; Versuch eines vollkommen consequenten Systems der Mathematik, vol. 4, Berlin, 1830, p. 136–137; Geist der Differential- und Integral-Rechnung, Erlangen, 1846, p. 51 ff.

† A. R. Forsyth, Theory of Functions of a Complex Variable, Cambridge, 1893, p. 27.

‡ C. Jordan, Cours d'Analyse, 3. Edition, vol. I, 1909, p. 34, 35.

§ V. Volterra, Giornale di matematiche (Battaglini), vol. 19, 1881, p. 340.

‖ G. Peano, Formulaire mathématique, vol. IV, Turin, 1903, p. 178.

¶ Lhuilier, Exposition élémentaire des principes des calculs supérieurs, Berlin, 1786, p. 24, 31.

** Œuvres mathématique du Citoyen Carnot, à Basle, 1797, p. 191–198, 168.

†† John Brinkley in Trans. Royal Irish Academy, vol. XIII, Dublin, 1818, p. 30.

‡‡ A. L. Cauchy, Cours d'analyse, Paris, 1821, p. 13, 14.

The period in "lim." was gradually dropped and "lim" came to be the recognized form.

K. Weierstrass. In the nineteenth century many writers felt the need of a symbolism indicating also the limit which the independent variable approaches. The notation which has been used widely in Europe is "Lim$_{x\ a}$," to express "the limit as x approaches a." It is found in the papers of Karl Weierstrass, who in 1841* wrote "lim" and in 1854† wrote "Lim.$_{n\ \infty} p_n = \infty$." W. R. Hamilton in 1853‡ used "$\lim_{n=\infty}\{\ \ \}^{\varepsilon=1}$."

J. E. Oliver. In the United States the symbol $\doteq$ has been used extensively. It is due to James E. Oliver of Cornell University,§ and appeared in print in 1880 in W. E. Byerly's calculus. This symbol was used when it became desirable to indicate that the variable was not allowed to reach its limit, though in recent years it has been used, also when no such restriction seemed intended. Oliver¶ himself used it in print in 1888. He says: "Using $\doteq$ in the sense 'Approaches in value to, as a limit', $U \doteq_{\Delta x} C$ would naturally mean 'U approaches C as a limit, when Δx approaches its limit (viz. 0).'" Again he says: "Between its meaning as first used by me, 'Is nearly equal to,' and Byerly's modification, 'Approaches as a limit', and perhaps yet other useful shades of interpretation, the context will sufficiently determine." To mark the degree of approximation, Oliver writes $\doteq^1$, $\doteq^2$, ... $\doteq^n$, so that "$U \doteq^n hV$" would mean that $U - V$ or $(U-V)/hV$ is of at least the nth order of smallness or negligeability, h being of the first order. James Pierpont** writes $a_n \doteq l$ for $\lim a_n = l$.

W. H. Echols†† of the University of Virginia advocated the use of the symbol £ for the term "limit" and the symbol $(=)$ in preference to $\doteq$. E. W. Hobson‡‡ uses the sign $\sim$ to indicate approach to a limit.

* K. Weierstrass, Mathematische Werke, 1. Band, Berlin, 1894, p. 60.

† Crelle's Journal, vol. 51, 1856, p. 27; Mathematische Werke, 1. Band, p. 155.

‡ W. R. Hamilton, Lectures on Quaternions, Dublin, 1853, p. 569.

§ See D. A. Murray, An Elementary Course in the Integral Calculus, New York, 1898, p. 4, Note.

‖ W. E. Byerly, Elements of the Differential Calculus, Boston, 1880, p. 7.

¶ Annals of Mathematics, Charlottesville, Va., vol. 4, 1888, p. 187, 188. The symbol is used also in Oliver, Wait and Jones' Algebra, Ithaca, 1887, p. 129, 161.

** James Pierpont, Theory of Functions of Real Variables, New York, vol. I, 1905, p. 25.

†† W. H. Echols, Elementary Text-book on the Differential and Integral Calculus, New York, 1902, Preface, also p. 5. We quote from the Preface: "... the use of the 'English pound' mark for the symbol of 'passing to the limit' is so suggestive and characteristic that its convenience has induced me to employ it in the text, particularly as it has been frequently used for this purpose here and there in the mathematical journals. The use of the 'parenthetical equality' sign $(=)$ to mean 'converging to' has appeared more convenient .... than the dotted equality $\doteq$, which has sometimes been used in American texts."

‡‡ E. W. Hobson, Theory of Functions of a Real Variable, 2d Ed., Cambridge, 1921.

J. G. Leathem. In England an arrow has been used in recent years in place of $=$. In 1905 J. G. Leathem* of St. John's College, Cambridge, introduced $\longrightarrow$ to indicate continuous passage to a limit and he suggested later (1912) that a dotted arrow might appropriately represent a saltatory approach to a limit. The full arrow is meeting with general adoption nearly every where. G. H. Hardy† writes $\lim_{n\to\infty}(1/n) = 0$ and introduces the $\longrightarrow$ with the remark, "I have followed Mr. J. G. Leathem and Mr. T. J. I'A. Bromwich in always writing $\lim_{n\to\infty}$, $\lim_{x\to\infty}$, $\lim_{x\to a}$, and not $\lim_{n=\infty}$, $\lim_{x=\infty}$, $\lim_{x=a}$. This change seems to me one of considerable importance, especially when '∞' is the 'limiting value.' I believe that to write $n = \infty$, $x = \infty$ (as if anything ever were 'equal to infinity'), however convenient it may be at a later stage, is in the early stages of mathematical training to go out of one's way to encourage incoherence and confusion of thought concerning the fundamental ideas of analysis." Sometimes the "lim" is omitted,‡ as in $\frac{\Delta u}{\Delta x} \longrightarrow u'$. F. S. Carey,§ following a suggestion made by James Mercer, uses for monotone sequences an arrow with a single barb, either upper or lower; thus $\longrightarrow$ "tends up to" the limit, $\longrightarrow$ "tends down to" the limit.

G. L. Dirichlet. G. Lejeune Dirichlet‖ in 1837 introduced a symbolism which we give in his own words: "Es wird im Folgenden nöthig sein, diese beiden Werthe von $\varphi(\beta)$ zu unterscheiden, und wir werden sie durch $\varphi(\beta - 0)$ und $\varphi(\beta + 0)$ bezeichnen." We have here the first suggestion of our present notation, "$\lim x = a + 0$," when x converges to a by taking only values greater than a, and "$\lim x = a - 0$," when x converges to a by taking values all less than a.

A. Pringsheim. The upper and the lower limit of y as a function of x was designated by M. Pasch¶ of Giessen as "$\limsup_{x=\infty} y$" and "$\liminf_{x=\infty} y$." For the same purpose A. Pringsheim of Munich introduced the notations $\overline{\lim_{\nu=+\infty}} a_\nu = A$ and $\underline{\lim_{\nu=+\infty}} a_\nu = \alpha$ to denote, respectively, the upper and lower limits of the variable a_ν,** and $\overline{\lim_{\nu=+\infty}} a_\nu$ to indicate that one may take at pleasure, either the upper or the lower limit.

* J. G. Leathem, Surface Integrals used in Physics, 2. Ed., Cambridge, 1913. Preface.

† G. H. Hardy, Course of Pure Mathematics, 1908, p. 116. See also the Preface.

‡ See for example, Maurice Laboureur, Cours de calcul algébrique Différentiel et intégral, Paris et Liege, 1913, p. 102.

§ F. S. Carey, Infinitesimal Calculus, London, 1919. Preface.

‖ Repertorium der Physik, vol. I, Berlin, 1837, p. 170.

¶ M. Pasch, Mathematische Annalen, vol. 30, 1887, p. 134.

** Sitzungsberichte Akad. München, Math.-phys. Classe, vol. 28, 1898, p. 62. See also Encyclopédie des scienc. math., Tome I, vol. I, p. 189, note 224; p. 232, note 45.

Paul du Bois-Reymond* of the University of Freiburg in Baden introduced the notations $f(x) \succ g(x)$, $f(x) \sim g(x)$, $f(x) \prec g(x)$, as equivalents to the formulas

$$\lim \frac{f(x)}{g(x)} = \infty, \quad \lim \frac{f(x)}{g(x)} \text{ is finite}, \quad \lim \frac{f(x)}{g(x)} = 0.$$

These symbols were used by A. Pringsheim† who in 1890 added $f_1(n) \cong a f_2(n)$ for $\lim \dfrac{f_1(n)}{f_2(n)} = a$.

L. Scheffer. The four values (viz. Du Bois-Reymond's Unbestimmtheitsgrenzen) which the difference-quotient $\dfrac{\Delta y}{\Delta x}$ approaches, as Δx approaches zero (y being a single-valued function of x) were designated by Dini‡ by the symbols $\Lambda, \lambda, \Lambda', \lambda'$: the first two being the right-hand upper and lower limits, the last two the left-hand upper and lower limits. Ludwig Scheffer§ used in 1884 for these four limits, respectively, the symbols D^+, D_+, D^-, D_-, which were modified by M. Pasch in 1887 to D^+, $-D$, D^-, $-D$.

G. Peano. The infinite values, when x approaches zero, of $\lim\left(\left(\sin \frac{1}{x}\right)\right)$, cited by Cauchy in 1821, constitute a class, for the designation of which G. Peano¶ introduced the sign Lm, in place of the ordinary lim. By Lm x Peano means "les valeurs limites de x" or "la classe limite de x." The need of this new abbreviation is an indication that, since the time when Cauchy began to publish, a new concept, the idea of classes, has become prominent in mathematics. By Lm (f, u, x) Peano** designates" les valeurs limites de la fonction f, lorsque la variable, en variant dans la classe u, tend vers x."

W. H. Young. The consideration of the plurality of limits has brought forth other special notations. Thus W. H. Young†† states in 1910: "We use the notation $\underset{x=a}{\mathrm{Llt}}\, f(x)$ to denote the set of all the limits of $f(x)$ at a point a, while, if it is known that there is an unique limit, we write $\underset{x=a}{\mathrm{Lt}}\, f(x)$ for that limit. . . . If there are two independent variables x and y, the limits at (a, b) . . . are called *double limits*." He considers "*repeated limits* of $f(x, y)$ first with respect to x and then with respect to y, and written $\underset{y=b}{\mathrm{Llt}}\, \underset{x=a}{\mathrm{Lt}}\, f(x, y)$." Similarly he considers repeated limits, first with respect to y and then with respect to x.

* Paul du Bois-Reymond in Annali di mathematica pura ed applicata. Diretti da F. Brioschi e L. Cremona. Milano, Serie II. Tomo IV. 1870-71. p. 339.

† A. Pringsheim in Mathematische Annalen, vol. 35, 1890, p. 302.

‡ Dini, Fondamenti per la teorica delle funzioni di variabili reali, 1878, Art. 145, p. 190.

§ L. Scheffer, Acta Mathematica, vol. 5, Stockholm, 1884, p. 52, 53.

M. Pasch, Mathematische Annalen, vol. 30, 1887, p. 135.

¶ G. Peano, Formulaire mathématique, Tome IV, Turin, 1903, p. 148.

** Loc. cit., p. 164.

†† W. H. Young, The Fundamental Theorems of the Differential Calculus, Cambridge University Press, 1910, p. 3, 4.

5. Concluding Observations.

In considering the history of the calculus, the view advanced by Moritz Cantor presses upon the mind of the reader with compelling force. Cantor says:* "We have felt that we must place the main emphasis upon the notation. This is in accordance with the opinion which we have expressed repeatedly that, even before Newton and Leibniz, the consideration of infinitesimals had proceeded so far that a suitable notation was essential before any marked progress could be made."

Our survey of calculus notations indicates that this need was met, but met conservatively. There was no attempt to represent all reasoning in the calculus by specialized short-hand signs so as to introduce a distinct sign-language which would exclude those of ordinary written or printed words. There was no attempt to restrict the exposition of theory and application of the calculus to ideographs. Quite the contrary. Symbols were not generally introduced, until their need had become imperative. Witness, for instance, the great hesitancy in the general acceptance of a special notation for the partial derivative.

It is evident that a sign, to be successful, must possess several qualifications: It must suggest clearly and definitely the concept and operation which it is intended to represent; it must possess adaptability to new developments in the science; it must be brief, easy to write and easy to print. The number of desirable mathematical symbols that are available is small. The letters of the alphabet, the dot, the comma, the stroke, the bar, constitute the main source of supply. In the survey made in this paper, it was noticed that the forms of the fourth letter of the alphabet, d, D and ∂ were in heavy demand. This arose from the fact that the words "difference," "differential," "derivative," "derivation" all began with that letter. A whole century passed before any general agreement was reached among mathematicians of different countries, on the specific use which should be assigned to each form.

The query naturally arises, could international committees have expedited the agreement? Perhaps the international association for the promotion of vector analysis will afford an indication of what may be expected from such agencies.

An interesting feature in our survey is the vitality exhibited by the notation $\frac{dy}{dx}$ for derivatives. Typographically not specially desirable, the symbol nevertheless commands at the present time a wider adoption than any of its many rivals. Foremost among the reasons for this is the flexibility of

* M. Cantor, Vorlesungen über Geschichte der Mathematik, vol. III, 2. Edition, Leipzig, 1901, p. 167.

the symbol, the ease with which one passes from the derivative to the differential by the application of simple algebraic processes, the intuitional suggestion to the mind of the reader of the characteristic right triangle which has dy and dx as the two perpendicular sides. These symbols readily call to mind ideas which reach to the very heart of geometric and mechanical applications of the calculus.

For integration the symbol $\int$ has had practically no rival. It easily adapted itself to the need of marking the limits of integration in definite integrals. When one considers the contributions that Leibniz has made to the notation of the calculus, and of mathematics in general, one beholds in him the most successful and influential builder of symbolic notation that the science ever has had.

"March on, symbolic host! with step sublime,
Up to the flaming bounds of Space and Time."
JAMES CLERK MAXWELL.

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NEW APPLICATIONS OF A FUNDAMENTAL THEOREM OF SUBSTITUTION GROUPS.

By G. A. Miller.

1. **Introduction.** The fundamental theorem in question may be stated as follows. If the cycle $a_1 \cdots a_l$ appears in at least one substitution of the group G, and if we count this cycle and its conjugates under G as many times as each of them appears in different substitutions of G, then the total number of letters in these cycles is equal to the order of G.* For instance, the group of the square is composed of the following substitutions:

$$1,\ ac\cdot bd,\ ac,\ bd,\ ab\cdot cd,\ ad\cdot bc,\ abcd,\ adcb.$$

The two cycles ac, bd constitute a complete set of conjugates under this group, and each of them appears in two substitutions. The total number of letters involved in these cycles when each of them is counted twice is 8, which is also the order of the group under consideration. The cycles ab, ad, bc, cd also constitute a complete set of conjugates and each of them appears only once. The total number of letters involved is again 8. Similarly, the cycles $abcd$ and $adcb$ are conjugate and involve 8 letters.

If G contains an invariant subgroup H, then all the conjugates under G of an arbitrary cycle of H appear in H. If all of these cycles are also conjugate under H, then they must appear also in substitutions of G which are not found in H. In general these cycles will appear in k sets of conjugates under H. If the index of H under G exceeds k, then these cycles must appear also in substitutions of G which are not found in H. The number of sets of conjugate substitutions contained in a group may be greater than, equal to, or less than the number of the sets of the conjugate cycles. The latter number may be determined from the following theorem: *In every substitution group of degree n the number of sets of conjugate cycles is $n-l$, where l represents the number of systems of intransitivity of the group.*

This theorem results directly from the fundamental theorem noted above and the theorem that the average number of letters in all the substitutions of a group of degree n is $n-l$, l being the number of its systems of intran-

* Cf. G. A. Miller, Amer. Journ. Math., vol. 44 (1922), p. 122.

sitivity. A complete set of conjugate cycles of a transitive group of degree n involves each letter of this group at least once. If it involves it only once, the number of times each of these cycles appears in substitutions of this group is equal to the order of one of its subgroups composed of all its substitutions which omit a given letter. This is evidently the largest number of times the same cycle can appear in the substitutions of a transitive group. In general, *the number of times the same cycle of a transitive group appears in the substitutions of this group is a divisor of the quotient obtained by dividing its order by its degree.* A necessary and sufficient condition that this number is equal to this quotient is that the conjugates of the cycle form a substitution which is commutative with every substitution of the transitive group.

If a group of order $p(p+1)$, p being a prime number, contains $p+1$ subgroups of order p, it can be represented as a transitive group of degree $p+1$ and each of its subgroups of order p is transformed into itself by only p substitutions. Hence such a group contains $p-1$ sets of conjugate cycles of order p, and it can therefore contain only one other set of conjugate cycles. These cycles must therefore be of order 2 and hence $p+1$ must be of the form 2^α. Moreover, the subgroup of order $p+1$ must be abelian and of type $(1, 1, 1, \ldots)$. In particular, there is no group of order 30 and degree 6. The degree of every substitution group is at least equal to the number of the different orders of its cycles increased by the number of its systems of intransitivity.

As a very elementary illustration of the usefulness of the theorem noted in the opening sentence, we shall employ it to prove that the alternating group of degree 4, or the tetrahedral group, contains no subgroup of order 6. It is obvious that this alternating group contains 3 substitutions of the form $ab \cdot cd$ and 8 of the form abc, besides the identity. If a subgroup of order 6 existed, it could not contain only one substitution of the form $ab \cdot cd$ since the number of letters in its cycles is not a multiple of 6. If it contained two such substitutions, it would also contain their product, which is the third. This is impossible because the order of every subgroup divides the order of the group. Hence a subgroup of order 6 would have to contain five substitutions of the form abc, besides the identity. This is again impossible since the number of letters in these five cycles would not be a multiple of 6. Hence this alternating group does not involve any subgroup of order 6.

In this proof little use was made of other well known theorems relating thereto in order to exhibit more clearly the power of the particular theorem under consideration. In what follows we shall, however, make use also of a few other well known theorems—such as Sylow's theorem and the theorem that the average number of letters in all the substitutions of a group of degree n is $n-l$, l being the number of its systems of intransitivity. The still

more fundamental theorem that the order of a group is divisible by the order of each of its subgroups was used above. The fact that the order of every transitive group of degree n is a multiple of n will also be frequently assumed. The usefulness of the theorem under consideration can perhaps be most easily seen in the determination of the possible substitution groups of low degrees and hence we begin with this subject.

2. **Groups of degree four.** Every intransitive group of degree 4 may be supposed to involve as constituents the following two groups of degree 2:

$$1,\ ab \qquad\qquad 1,\ cd.$$

Since the cycle ab is invariant under such a group, the total number of letters in this cycle, counted as many times as it appears, must equal the order of the group. As it contains only two letters, the number of substitutions which involve this cycle must equal the number which do not involve it. As the latter number cannot exceed 2, the order of the group cannot exceed 4. If this order is 4, the group is composed of the following substitutions:

$$1,\ cd,\ ab,\ ab\cdot cd.$$

When this order is 2, the group is obviously

$$1,\ ab\cdot cd.$$

Hence there are two and only two intransitive groups of degree 4.

It was proved in the introduction that the alternating group of degree 4 does not contain any subgroup of order 6, and from this it follows directly that the symmetric group of degree 4 contains only one subgroup of order 12. Hence there is one and only one group of each of the orders 12 and 24 on four letters. There is also only one such group of order 8 according to Sylow's theorem.

To illustrate the usefulness of the theorem noted at the close of the second paragraph of the introduction we shall employ it to prove directly that all the subgroups of order 8 contained in the symmetric group of degree 4 are conjugate. All these groups must have in common the subgroup of order 4 contained in the alternating group. Since they are transitive, they must involve three sets of conjugate cycles. Hence two cycles of order 2 must appear twice therein; and there are three subgroups of order 8 in the symmetric group

of degree 4. As these cycles of order 2 are conjugate under this symmetric group, the three subgroups of order 8 must also be conjugate.

It remains only to consider the possible transitive groups of order 4 on four letters. These groups must be regular. If such a group involves negative substitutions, it must be cyclic, and there is only one such group since the cycles of order 4 are conjugate under the symmetric group of degree 4. There is also only one group composed of positive substitutions since the alternating group of degree 4 contains only one subgroup of order 4. This therefore completes the determination of the seven possible substitution groups of degree 4 by means of the theorem under consideration.

3. **Groups of degree five.** Every intransitive group of degree 5 may be supposed to involve 1, ab as one transitive constituent group while the other constituent group is the symmetric or the alternating group on three letters. Half of the substitutions of such a group must involve ab since the cycle ab is invariant and the number of letters contained in it when counted for all the substitutions in which it appears is equal to the order of the group. The substitutions which do not contain it must constitute the alternating or the symmetric group of degree 3. Hence the order of such an intransitive group is either 6 or 12. When it is 6, there are two possible groups since the second constituent group may be either the alternating or the symmetric group on three letters. Hence there are three and only three intransitive groups of degree 5. Two of them are of order 6 and the third is of order 12.

Since the order of a transitive group of degree 5 is a multiple of 5 and a divisor of 120, the only numbers which need to be considered as possible orders are 5, 10, 15, 20, 30, 40, 60, 120. There can be no group of order 15 since the number of letters in the cycles of a subgroup of order 5 is 20 and hence there would have to be more than one such subgroup. If a group contains more than one Sylow subgroup of order 5, it must contain at least 6 such subgroups and hence at least 25 substitutions including the identity. If there were a group of order 30 or 40 on five letters, the number of letters on its cycles of order 5 would again have to exceed 20 and hence such a group would have 6 subgroups of order 5. As these would be conjugate, the order could not be 40. If the order were 30, there would have to be at least 10 cycles of the form abc since the number of letters in such cycles would be a multiple of 30. This is impossible since there would be at least 24 cycles of order 5.

It has now been proved that there is no group on 5 letters whose order is 15, 30 or 40. It remains therefore only to consider the possible groups of orders 5, 10, 20, 60, and 120. As there is no group of order 30, there can be only one group of order 60 and there is evidently only one group of order 120. There is only one group of order 5 by Sylow's theorem. In a group of order 20

the 4 cycles of order 5 would have to be conjugate since the number of letters found therein would be 20. As there are only 5 substitutions on these letters which transform such a cycle into itself, the group is completely determined by its subgroup of order 5 and hence there is only one such group. As this group contains only one subgroup of order 10, there is also only one group of the latter order on 5 letters. The transpositions in this group involve 20 letters and hence half of them are conjugate. A similar remark applies to the cycles of order 5. Hence there are five transitive and three intransitive groups of degree 5.

4. **Exponents modulo a power of a prime number.** Although the theorem under consideration is expressed in the language of substitution groups, it may be employed to furnish an easy proof of several theorems relating to the exponents to which various numbers belong with respect to a modulus which is a power of a prime number. In particular, it can be used to advantage in the study of the primitive roots of numbers which have such roots. In this work it is convenient to make use of the following elementary theorem: *If a regular abelian group H is extended by means of a substitution of prime order p which transforms H into itself and is commutative with each of the substitutions in a subgroup of H, then the extended group contains each cycle of this subgroup just p times.* This theorem follows directly from the fact that all the cycles contained in a substitution of a regular abelian group are conjugate under this group.

From the theorem noted in the preceding paragraph it follows directly that, if a regular cyclic group H of order p^m is extended by a substitution of order p on the letters of H which is commutative with the substitutions of order p^{m-1} and transforms H into itself, then the extended group of order p^{m+1} involves a subgroup of order p^m obtained by establishing an isomorphism between the p subgroups of order p^{m-1} contained in H. With the exception of the special case when $p = 2$ and $m = 2$, this extended group contains $p-1$ cyclic groups of order p^m which are not found in H and their generators involve all its cycles which do not also appear in H. In particular, it may be assumed that the extending substitution of order p is of degree $p^m - p^{m-1}$ and that the $p-1$ products obtained by multiplying it into the substitutions of order p contained in H are of the same degree. The group generated by these substitutions is of order p^2 and involves p^{m-1} systems of intransitivity.

From the preceding paragraph it results that the numbers which are of the form $1 + kp^{m-1}$, k being prime to p, belong to exponent p modulo p^m. It is easy to prove now that all the numbers of the form $1 + kp^\alpha$, $\alpha > 1$, belong to exponent $p^{m-\alpha}$. When p is odd, this is also true when $\alpha = 1$. To prove this theorem by means of the subgroup H it is only necessary to note that, when $\alpha = m-2$ and $m > 3$, then a substitution s which omits some of the letters

of degree 4. As these cycles of order 2 are conjugate under this symmetric group, the three subgroups of order 8 must also be conjugate.

It remains only to consider the possible transitive groups of order 4 on four letters. These groups must be regular. If such a group involves negative substitutions, it must be cyclic, and there is only one such group since the cycles of order 4 are conjugate under the symmetric group of degree 4. There is also only one group composed of positive substitutions since the alternating group of degree 4 contains only one subgroup of order 4. This therefore completes the determination of the seven possible substitution groups of degree 4 by means of the theorem under consideration.

3. **Groups of degree five.** Every intransitive group of degree 5 may be supposed to involve 1, ab as one transitive constituent group while the other constituent group is the symmetric or the alternating group on three letters. Half of the substitutions of such a group must involve ab since the cycle ab is invariant and the number of letters contained in it when counted for all the substitutions in which it appears is equal to the order of the group. The substitutions which do not contain it must constitute the alternating or the symmetric group of degree 3. Hence the order of such an intransitive group is either 6 or 12. When it is 6, there are two possible groups since the second constituent group may be either the alternating or the symmetric group on three letters. Hence there are three and only three intransitive groups of degree 5. Two of them are of order 6 and the third is of order 12.

Since the order of a transitive group of degree 5 is a multiple of 5 and a divisor of 120, the only numbers which need to be considered as possible orders are 5, 10, 15, 20, 30, 40, 60, 120. There can be no group of order 15 since the number of letters in the cycles of a subgroup of order 5 is 20 and hence there would have to be more than one such subgroup. If a group contains more than one Sylow subgroup of order 5, it must contain at least 6 such subgroups and hence at least 25 substitutions including the identity. If there were a group of order 30 or 40 on five letters, the number of letters on its cycles of order 5 would again have to exceed 20 and hence such a group would have 6 subgroups of order 5. As these would be conjugate, the order could not be 40. If the order were 30, there would have to be at least 10 cycles of the form abc since the number of letters in such cycles would be a multiple of 30. This is impossible since there would be at least 24 cycles of order 5.

It has now been proved that there is no group on 5 letters whose order is 15, 30 or 40. It remains therefore only to consider the possible groups of orders 5, 10, 20, 60, and 120. As there is no group of order 30, there can be only one group of order 60 and there is evidently only one group of order 120. There is only one group of order 5 by Sylow's theorem. In a group of order 20

the 4 cycles of order 5 would have to be conjugate since the number of letters found therein would be 20. As there are only 5 substitutions on these letters which transform such a cycle into itself, the group is completely determined by its subgroup of order 5 and hence there is only one such group. As this group contains only one subgroup of order 10, there is also only one group of the latter order on 5 letters. The transpositions in this group involve 20 letters and hence half of them are conjugate. A similar remark applies to the cycles of order 5. Hence there are five transitive and three intransitive groups of degree 5.

4. **Exponents modulo a power of a prime number.** Although the theorem under consideration is expressed in the language of substitution groups, it may be employed to furnish an easy proof of several theorems relating to the exponents to which various numbers belong with respect to a modulus which is a power of a prime number. In particular, it can be used to advantage in the study of the primitive roots of numbers which have such roots. In this work it is convenient to make use of the following elementary theorem: *If a regular abelian group H is extended by means of a substitution of prime order p which transforms H into itself and is commutative with each of the substitutions in a subgroup of H, then the extended group contains each cycle of this subgroup just p times.* This theorem follows directly from the fact that all the cycles contained in a substitution of a regular abelian group are conjugate under this group.

From the theorem noted in the preceding paragraph it follows directly that, if a regular cyclic group H of order p^m is extended by a substitution of order p on the letters of H which is commutative with the substitutions of order p^{m-1} and transforms H into itself, then the extended group of order p^{m-1} involves a subgroup of order p^m obtained by establishing an isomorphism between the p subgroups of order p^{m-1} contained in H. With the exception of the special case when $p = 2$ and $m = 2$, this extended group contains $p-1$ cyclic groups of order p^m which are not found in H and their generators involve all its cycles which do not also appear in H. In particular, it may be assumed that the extending substitution of order p is of degree $p^m - p^{m-1}$ and that the $p-1$ products obtained by multiplying it into the substitutions of order p contained in H are of the same degree. The group generated by these substitutions is of order p^2 and involves p^{m-1} systems of intransitivity.

From the preceding paragraph it results that the numbers which are of the form $1 + kp^{m-1}$, k being prime to p, belong to exponent p modulo p^m. It is easy to prove now that all the numbers of the form $1 + kp^\alpha$, $\alpha > 1$, belong to exponent $p^{m-\alpha}$. When p is odd, this is also true when $\alpha = 1$. To prove this theorem by means of the subgroup H it is only necessary to note that, when $\alpha = m-2$ and $m > 3$, then a substitution s which omits some of the letters

of H and transforms a generator of H into its $1+kp^{m-2}$ power must be of order p^2. In fact, s gives rise to commutators of order p^2 but of no higher order. When p is odd, s is obviously also of order p^2 when $m=3$ but this is not the case when $p=2$. The theorem in question results now directly from the properties of the quotient group with respect to subgroups of H.

The theorem noted in the first paragraph of this section can evidently be extended so as to apply to all of the co-sets with respect to H of a Sylow subgroup of the holomorph of H when this holomorph is represented as a substitution group on the letters of H. In particular, each of these co-sets involves cycles of order p contained in H, and all these cycles appear in the substitutions which omit at least one letter. Moreover, those which appear in one co-set are equally distributed among such substitutions appearing in the same co-set. If s transforms a generator of H into a power whose index is of the form $1+kp^a$, it must therefore involve p^a-p^{a-1} such cycles.

UNIVERSITY OF ILLINOIS.

GEODESIC REPRESENTATION BETWEEN RIEMANN SPACES.

By H. Levy and A. Bramley.

The problem of representing one Riemann space upon another so that the geodesics correspond was solved by Levi-Civita.* Consider two Riemann spaces whose linear elements are

$$ds^2 = g_{rs}\, dx^r\, dx^s, \tag{1}$$

and

$$dt^2 = b_{rs}\, dx^r\, dx^s; \tag{2}$$

then the relation between the arcs s and t of corresponding curves must be

$$\frac{dt}{ds} = \sqrt{b_{rs}\, \frac{dx^r}{ds}\, \frac{dx^s}{ds}}\,.$$

Levi-Civita, however, considered this problem from the dynamical point of view asking under what conditions two dynamical systems with no forces will have the same trajectories. If

$$x^i = f^i(t), \qquad x^i = \varphi^i(s) \qquad (i = 1, 2, \ldots n)$$

are the respective equations of a pair of trajectories of the two equivalent systems, there must be a relation between the parameters t and s under which

$$f^i(t) = \varphi^i(s) \qquad (i = 1, 2, \ldots n).$$

Levi-Civita assumed that the relation between t and s is of such a nature that

$$\frac{dt}{ds} = \frac{1}{f},$$

* Annali di Mat., vol. 24 (1896), p. 255.

where f can be expressed as a power series in $\dot{x} = dx/ds$ with coefficients functions of x alone, and that the value of f for $\dot{x} = 0$ is not identically zero in the x's; f may then be written

$$f = \mu\left(1 + c_r \dot{x}^r + \frac{1}{2} c_{rs} \dot{x}^r \dot{x}^s + \cdots\right).$$

The function $f = ds\,dt$, in the case of geodesic correspondence between Riemann spaces, must become

$$\frac{1}{\sqrt{h_{rs} \dfrac{dx^r}{ds} \dfrac{dx^s}{ds}}}.$$

Since this function cannot be expanded in the above manner, the proof cannot be justified although the results obtained are correct.

The geodesics of the Riemann space whose linear element is given by (2) are

$$\frac{d^2 x^i}{dt^2} + \Delta^i_{\alpha\beta} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt} = 0, \qquad (i = 1, 2, \ldots n), \tag{3}$$

where the functions $\Delta^i_{\alpha\beta}$ are the Christoffel symbols of the second kind,

$$\Delta^i_{\alpha\beta} = \frac{1}{2} b^{ij} \left\{ \frac{\partial b_{\alpha j}}{\partial x^\beta} + \frac{\partial b_{\beta j}}{\partial x^\alpha} - \frac{\partial b_{\alpha\beta}}{\partial x^j} \right\}$$

and b^{ij} is the cofactor of b_{ij} in the determinant $b = |b_{ij}|$ divided by b. Since we must have

$$\frac{ds}{dt} = \frac{1}{\sqrt{h_{rs} \dfrac{dx^r}{ds} \dfrac{dx^s}{ds}}},$$

the system of differential equations (3) referred to s, the arc of the Riemann space (1), as parameter becomes

$$h_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} \frac{d^2 x^i}{ds^2} - h_{jk} \frac{dx^i}{ds} \frac{dx^j}{ds} \frac{d^2 x^k}{ds^2} + \left(\Delta^i_{jk} - \frac{1}{2} \frac{\partial b_{jk}}{\partial x_l} \frac{dx^l}{ds} \frac{dx^i}{ds}\right) \frac{dx^j}{ds} \frac{dx^k}{ds} = 0 \qquad (i = 1, 2, \ldots n). \tag{4}$$

If the Riemann spaces (1) and (2) are in geodesic correspondence, the system of equations (4) must be equivalent to the system

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{jk} \frac{dx^j}{ds} \frac{dx^k}{ds} = 0, \tag{5}$$

the equations of the geodesics of the first space. The functions Γ^i_{jk} are the Christoffel symbols of the second kind in the first space.

Comparing (4) and (5) we obtain

$$b_{jk} (\Gamma^i_{lm} - \Delta^i_{lm}) \frac{dx^j}{ds} \frac{dx^k}{ds} \frac{dx^l}{ds} \frac{dx^m}{ds} + \frac{1}{2} b_{jkl} \frac{dx^j}{ds} \frac{dx^k}{ds} \frac{dx^l}{ds} \frac{dx^i}{ds} = 0, \tag{6}$$

where

$$b_{jkl} = \frac{\partial b_{jk}}{\partial x^l} - \Gamma^h_{jl} b_{hk} - \Gamma^h_{kl} b_{hj}.$$

Since every geodesic of the one space corresponds to a geodesic in the other, equations (6) must hold identically in the derivatives. Equating coefficients we obtain the following equations:

$$\Gamma^i_{jk} - \Delta^i_{jk} = -\frac{1}{4} \left(\delta^i_j \frac{b_{kkk}}{b_{kk}} + \delta^i_k \frac{b_{jjj}}{b_{jj}} \right), \tag{7}$$

$$2 b_{jk} (\Gamma^i_{ii} - \Delta^i_{ii}) + 4 b_{ki} (\Gamma^i_{ij} - \Delta^i_{ij}) + 4 b_{ij} (\Gamma^i_{ik} - \Delta^i_{ik}) + b_{ijk} + b_{jki} + b_{kij} = 0,$$

where i, j, k = 1, 2, . . . n, not summed, and therefore

$$b_{ijk} + b_{jki} + b_{kij} = b_{jk} \frac{b_{iii}}{b_{ii}} + b_{ki} \frac{b_{jjj}}{b_{jj}} + b_{ij} \frac{b_{kkk}}{b_{kk}},$$

where

$$\delta^i_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}.$$

It is well known* that

$$\frac{\partial b_{ij}}{\partial x^k} - b_{hi} \Delta^h_{jk} - b_{hj} \Delta^h_{ik} = 0.$$

* Cf. Bianchi, Lezioni di Geom. Diff., vol. 1, p. 65.

Replacing Δ^h_{ik} by its value as given by equations (7), we obtain without difficulty

$$(9) \qquad b_{ijk} - \frac{1}{2} b_{ij} \frac{b_{kkk}}{b_{kk}} - \frac{1}{4} b_{ik} \frac{b_{jjj}}{b_{jj}} - \frac{1}{4} b_{jk} \frac{b_{iii}}{b_{ii}} = 0. \quad i, j, k = 1, 2, \ldots n,$$

where $i, j, k = 1, 2, \ldots n$, not summed. Permuting i, j, k in (9) and adding we find that equations (8) are satisfied.

Eisenhart* has shown that, when the functions Γ^i_{jk} and Δ^i_{jk} of the two Riemann spaces satisfy equations of the kind

$$(10) \qquad \Gamma^i_{jk} - \Delta^i_{jk} = \delta^i_j \varphi_k + \delta^i_k \varphi_j,$$

φ_j is a gradient so that

$$\varphi_j = \frac{\partial \varphi}{\partial x^j}.$$

Hence from (7) we may write

$$\frac{b_{kkk}}{b_{kk}} = -2 \frac{\partial}{\partial x^k} \log \mu.$$

Because of (9), equations (7) and (8) may be replaced by

$$(11) \qquad \mu b_{ijk} + b_{ij} \mu_k + \frac{1}{2} (b_{ik} \mu_j + b_{jk} \mu_i) = 0,$$

where

$$\mu_i = \frac{\partial \mu}{\partial x^i}, \qquad (i = 1, 2, \ldots n).$$

Equations (11) are identical with those obtained by Levi-Civita† as necessary conditions that two Riemann spaces be in geodesic correspondence. From them he obtains explicit forms for the linear elements of the two spaces. Equations (10) are necessary and sufficient conditions that, in the geometry of paths, two spaces have the same paths.‡ If the spaces are Riemannian, we have the explicit form for the vector φ_j

$$\varphi_j = -\frac{1}{4} \frac{b_{jjj}}{b_{jj}} \qquad (j = 1, 2, \ldots n).$$

* Proc. Nat. Acad. Sci., vol. 8, p. 237.

† Loc. cit., p. 275.

‡ Eisenhart, loc. cit., p. 234.

ON THE RESIDUES OF FIGURATE NUMBERS.*

By Oliver E. Glenn.

1. The elementary notion of figurate numbers as integers of a sequence whose nth term is the sum of n terms of another sequence, all of the numbers of the first series employed being unity, is extended in this paper according to the following definition. Let S be any series of integers which proceeds according to any fixed law

$$S(a):\ a_1, a_2, \ldots, a_r, \ldots \infty,$$

and let these integers be considered as the initial numbers, respectively, of appropriate orders of differences in a table of differences. This table will be triangular in form and infinite in extent, viz.,

$$\begin{array}{lllll}
 & \cdots & \cdots & \cdots & \cdots \\
 & a_4 & a_3 + a_4 & a_2 + 2a_3 + a_4 & a_1 + 3a_2 + 3a_3 + a_4 \\
P_1: & a_3 & a_2 + a_3 & a_1 + 2a_2 + a_3 & \\
 & a_2 & a_1 + a_2 & & \\
 & a_1 & & &
\end{array}$$

The general element, the one in the nth column and the rth row, is

$$(1_1) \qquad \binom{n}{r} = \binom{n-1}{0} a_{r-n+1} + \binom{n-1}{1} a_{r-n+2} + \cdots + \binom{n-1}{n-1} a_r.$$

* Presented to the American Mathematical Society, April, 1922; December, 1922. Cf. Dickson, History of the theory of numbers, vol. 1, chap. 9, vol. 2, chap. 1; also papers listed in index of vol. 1 under "Differences." The relation between these and the present paper is remote except, in connection with § 6, cf. Dickson, Annals of Math. (1), vol. 2 (1896), p. 75, and Quart. Journ. Math., vol. 33 (1902), p. 381, Glaisher, Quart. Journ. Math., vol. 30 (1899), p. 361, Carmichael, On sequences of integers defined by recurrence relations, Quart. Journ. Math., vol. 48 (1920), p. 343.

which will sometimes be abbreviated

$$\left(\frac{u}{v}\right) = \begin{pmatrix} \alpha_{v-u+1},\ \alpha_{v-u+2},\ \ldots\ \alpha_v \\ u,\ v \end{pmatrix}.$$

We are to study the properties of the figurate numbers P_1 by means of the plexus P of residues of the numbers in P_1 with regard to the modulus p, a prime. Reference is continually made to interesting particular cases of P and to geometric aids in the study of the numbers of the plexus.

When the initial series $S(\alpha)$ is

(1) S: $5n_1+4,\ 5n_2+3,\ 5n_3+2,\ 5n_4+1,\ 5n_5,\ 5n_6+4,\ 5n_7+3, \ldots.$

where $n_1, n_2, \ldots$ are arbitrary positive integers and $p = 5$, P takes the following form:

. . . . . .

3 2 1 1 **0** 1 4 2 2 **0** 2 3

4 4 **0** 4 1 3 3 **0** 3 2 1

0 1 4 2 2 **0** 2 3 4 4

1 3 3 **0** 3 2 1 1 **0**

2 **0** 2 3 4 4 **0** 4

3 2 1 1 **0** 1 4

4 4 **0** 4 1 3

0 1 4 2 2

1 3 3 **0**

2 **0** 2

3 2

4

Fig. 1.

The zero elements form the vertices of a quadrangular net. There is a general theorem of which this is an example. Consider also another example noting especially the configurations of the elements of the plexus which are congruent to zero modulo $p(=3)$:

S: $3n_1$, $3n_2+1$, $3n_3+2$, $3n_4+1$, $3n_5+2$, $3n_6$, $3n_7+2$,
$3n_8$, $3n_9+1$, $3n_{10}$, $3n_{11}+1$, $3n_{12}+2$, ...

. . . . . .

0 2 1 2 2 1 2 0 0 0 1 2 1 1 2 1 0 0 0 2 1 2 2 1 2 0
2 2 1 0 2 1 1 0 0 1 1 2 0 1 2 2 0 0 2 2 1 0 2 1 1
0 2 2 2 2 0 2 0 1 0 1 1 1 1 0 1 0 2 0 2 2 2 2 0
2 0 0 0 1 2 1 1 2 1 0 0 0 2 1 2 2 1 2 0 0 0 1
1 0 0 1 1 2 0 1 2 2 0 0 2 2 1 0 2 1 1 0 0 1
2 0 1 0 1 1 1 1 0 1 0 2 0 2 2 2 2 0 2 0 1
1 1 2 1 0 0 0 2 1 2 2 1 2 0 0 0 1 2 1 1
0 1 2 2 0 0 2 2 1 0 2 1 1 0 0 1 1 2 0
1 1 0 1 0 2 0 2 2 2 2 0 2 0 1 0 1 1
0 2 1 2 2 1 2 0 0 0 1 2 1 1 2 1 0
2 2 1 0 2 1 1 0 0 1 1 2 0 1 2 2
0 2 2 2 2 0 2 0 1 0 1 1 1 1 0
2 0 0 0 1 2 1 1 2 1 0 0 0 2
1 0 0 1 1 2 0 1 2 2 0 0 2
2 0 1 0 1 1 1 1 0 1 0 2
1 1 2 1 0 0 0 2 1 2 2
0 1 2 2 0 0 2 2 1 0
1 1 0 1 0 2 0 2 2
0 2 1 2 2 1 2 0
2 2 1 0 2 1 1
0 2 2 2 2 0
2 0 0 0 1
1 0 0 1
2 0 1
1 1
0

Fig. 2.

2. **Equivalences.** If we delete from P the series $S(\alpha)$ and the first row of residues parallel to the hypotenuse, what is left is a plexus of figurate numbers which proceeds by the evident additive process from the second column of P as an initial series $S(\alpha')$. Similarly by deleting the first r additive series $S(\alpha)$, $S(\alpha')$, ... $S(\alpha^{(r-1)})$ and the first r rows parallel to the hypotenuse

we obtain a plexus P' which can be regarded as having originated from $S(a^{(r)})$. We say that P' is additively equivalent to P.

The plexus $P^{(z)}$ derived from the sequence $S(z\,a)$, z being a fixed integer, is defined to be in multiplicative equivalence with P. The null elements (mod. p) of P and $P^{(z)}$ are in identical positions.

3. **The sieve of Eratosthenes.*** The method in the theory of numbers known by the name *cribrum Eratosthenis*, fundamental in researches on the distribution of primes in the linear sequence of positive integers, can be extended to two dimensions in connection with the problem of the distribution of primes in a table P_1 of figurate numbers.

Form the plexus P for each prime 2, 3, 5, ..., superimposing these residue tables upon each other in such a way that all residues modulis 2, 3, 5, ... of any element e of P_1 are written within one square or mesh of P. To find all prime elements e of P_1 less than a number l, proceed up the series of primes 2, 3, 5, ... until the first remaining number directly following the prime whose plexus was last formed has its square $> l$. Then every number of P_1 less than l whose mesh contains no zero is a prime. Every number $> l$ whose mesh contains no zero is at least relatively prime to all prime numbers used in the formation of the sieve.

Consider the series formed by assigning to each number n_i in (1) the value i of its subscript:

$$S(n):\ 3,\ 7,\ 11,\ 13,\ 17,\ 18,\ 23,\ 24,\ 28,\ 30,\ 34,\ 38,\ 40,\ 44,\ 45,\ 50,\ \ldots.$$

The complete list of primes which occur among the 136 numbers which make up the first 16 rows of the plexus P_1 whose initial series is $S(n)$ is the following:

$$\left(\frac{1}{1}\right) = 3,\ \left(\frac{1}{2}\right) = 7,\ \left(\frac{1}{3}\right) = 11,\ \left(\frac{1}{4}\right) = 13,\ \left(\frac{1}{5}\right) = 17,\ \left(\frac{1}{7}\right) = 23,$$

$$\left(\frac{2}{7}\right) = \binom{1}{0}18 + \binom{1}{1}23 = 41, \qquad \left(\frac{2}{8}\right) = \binom{1}{0}23 + \binom{1}{1}24 = 47,$$

$$\left(\frac{2}{15}\right) = \binom{1}{0}44 + \binom{1}{1}45 = 89, \qquad \left(\frac{3}{15}\right) = \binom{2}{0}40 + \binom{2}{1}44 + \binom{2}{2}45 = 173,$$

$$\left(\frac{5}{15}\right) = \binom{4}{0}34 + \binom{4}{1}38 + \binom{4}{2}40 + \binom{4}{3}44 + \binom{4}{4}45 = 647,$$

* Eratosthenes, Third century B. C. Tschebyscheff (Theorie der Congruenzen, in Russian 1849, in German 1889, p. 4) gives the dates 276 (or 275) to 194 B. C.

$$\binom{8}{13} = \binom{7}{0}18 + \binom{7}{1}23 + \binom{7}{2}24 + \binom{7}{3}28 + \binom{7}{4}30 + \binom{7}{5}34 + \binom{7}{6}38 + \binom{7}{7}40 = 3733,$$

$$\binom{9}{9} = \binom{8}{0}3 + \binom{8}{1}7 + \binom{8}{2}11 + \binom{8}{3}13 + \binom{8}{4}17 + \binom{8}{5}18 + \binom{8}{6}23 + \binom{8}{7}24 + \binom{8}{8}28 = 4157.$$

The numbers along the diagonal of such a table increase in magnitude rapidly; for example, $\binom{14}{14} = 191381\ (= 97 \cdot 1973)$. In the formation of the sieve, however, only residues are written.

4. **Drift lines.** Suppose α_i to be the element of general position of the series $S(\alpha)$ and draw a straight line through α_i and the element number $i-1$ of the third column. The elements which this line intersects when it is produced to the opposite bound of the plexus are the following:

$$\binom{2r+1}{i+r} = \binom{2r}{0}\alpha_{i-r} + \binom{2r}{1}\alpha_{i-r+1} + \cdots + \binom{2r}{2r}\alpha_{i+r}, \qquad (r = 0, \ldots, i-1). \tag{2}$$

If these numbers are all congruent to zero modulo p, the line is said to be a drift line.

Assume that the series $S(\alpha)$, reduced modulo p, is periodic, a complete period extending from α_i to α_{i+2s} inclusive. Assign to the series, moreover, the property

$$\alpha_{i+(2s-1)r+j} + \alpha_{i+(2s-1)r-j} \equiv 0 (\mathrm{mod}.\, p), \tag{3}$$

$j = 0, \ldots, s$; $r = 0, \ldots, \infty$. Then $\alpha_{i+(2s-1)r} \equiv 0 (\mathrm{mod}.\, p)$, $(p > 2)$, and P_1 is defined to be skew-symmetrical modulo p.

THEOREM. *If the plexus P_1 corresponding to a chosen series $S(\alpha)$ is skew-symmetrical modulo p, it is divided into strips of finite length and equal width by parallel oblique drift lines.*

For, since $\binom{2r}{\sigma} = \binom{2r}{2r-\sigma}$ $(\sigma = 0, \ldots, 2r)$, we have

$$\binom{2r+1}{i+r} \equiv 0 (\mathrm{mod}.\, p), \qquad (r = 0, \ldots, i+(2s+1)r-1),$$

on the line determined by each number $\alpha_{i+(2s+1)r}$.

THEOREM. *Midway between two consecutive drift lines of P_1 (modulo p), intersecting zero elements of $S(\alpha)$, there lies a drift line (secondary) which passes between two elements of $S(\alpha)$ but intersects a zero element of the first additive sequence $S(\alpha')$.*

The elements of $S(\alpha)$ included between drift lines number $r+1$ and $r+2$ are

$$\alpha_{i+(2s+1)r+1},\ \alpha_{i+(2s+1)r+2},\ \ldots,\ \alpha_{i+(2s+1)(r+1)-1},$$

whereas the elements of $S(\alpha')$ lying between the same pair of drift lines are represented by the following table:

$\alpha'_{i+(2s+1)r}$	$\alpha'_{i+(2s+1)r+1}$	$\cdots$	$\alpha'_{i+(2s+1)(r+1)-1}$
$\alpha_{i+(2s+1)r+1}$	$\alpha_{i+(2s+1)r+1}+\alpha_{i+(2s+1)r+2}$	$\cdots$	$\alpha_{i+(2s+1)(r+1)-1}+\alpha_{i+(2s+1)(r+1)}$

Hence we have

$$\alpha'_{i+(2s+1)r+s-j}+\alpha'_{i+(2s+1)r+s-j}\quad\alpha_{i+(2s+1)r+s-j}+\alpha_{i+(2s+1)r+s-j+1}$$

$$+\alpha_{i+(2s+1)r+s-j}+\alpha_{i+(2s+1)r+s-j+1}\equiv 0\ (\mathrm{mod.}\ p).$$

The latter congruence is an immediate consequence of (3). Thus the sequence $S(\alpha')$ possesses the property of skew symmetry, the zero elements (analogous to $\alpha_{i+(2s+1)r}$) being $\alpha'_{i+(2s+1)r+s}\equiv 0\ (\mathrm{mod.}\ p)$. The plexus P'', constructed from the first additive sequence $S(\alpha')$ and additively equivalent to P, has drift lines which intersect the null elements $\alpha'_{i+(2s+1)r+s}$. Moreover, since the arithmetic mean of $i+(2s+1)r$ and $i+(2s+1)(r+1)-1$ is $i+(2s+1)r+s$, the drift lines of P'' bisect the intervals between those of P. This is what we were to prove.

5. **Special sequences.** Let ϱ be any least residue modulo p and consider the following series $S_1(\alpha)$

$$S_1(\alpha):\ pn_1+\varrho,\ pn_2+2\varrho,\ pn_3+3\varrho,\ \ldots,\ pn_r+r\varrho,\ \ldots\ \infty,$$

where $n_1, n_2, \ldots$ are arbitrarily selected positive integers. The plexus P, corresponding, is the most general multiplicative equivalent modulo p of the plexus derived from the sequence of positive integers, i. e.

$$S(\alpha):\ 1, 2, \ldots, p-1, 0, p+1, \ldots 2p-1, 0, 2p+1, \ldots\ \infty.$$

Hence R is skew symmetric. We have for additive sequences

$S(\alpha')$: 3, 5, ..., p, ..., $2p-1$, $2p+1$, 3, 5, ...,

$3p$, ... $4p-1$, $4p+1$, 3, ... ∞.

$S(\alpha'')$: 4.2, 4.3, ..., $4(p-1)$, 0, $4(p+1)$, ... $4(2p-1)$, 0, $4(2p+1)$, ... ∞.

THEOREM. *The null elements (mod. p) of the plexus R are the vertices of a quadrangular net each mesh being a parallelogram.*

This theorem follows from the fact that the only zeros (mod. p) are those upon drift lines and these are at the vertices of a quadrangular net, two columns in width and $\frac{1}{2}(p+1)$ rows in length, whose sides are drift lines and oblique transversals; facts which are evident from the sequences $S(\alpha')$, $S(\alpha'')$.

6. **Factors of combinatory numbers.** Not only is the divisibility of figurate numbers by a prime p conditioned by that of the combinatory ${}_nC_r$, but the numbers $\binom{2\iota}{\iota}$ ($\iota = 0, \ldots, i-1$), themselves, are found as elements upon the line of skew symmetry in P in the case where $\alpha_i \equiv 1 \pmod{p}$ and all other numbers of $S(\alpha)$ are congruent to zero (cf. (2)). Most of the essential theory of the present paragraph is contained in the following two theorems.

THEOREM. *If any number of the set $\binom{p^\alpha+\beta}{j}$ ($\beta = 0, \ldots p-1$; $j = 0, 1, \ldots, p^\alpha+\beta$), is zero modulo p, being divisible by p^γ and not by $p^{\gamma+1}$, then the element $\binom{p^{\alpha-\delta}+\beta}{j}$ is divisible by $p^{\gamma-\delta}$ and not by $p^{\gamma-\delta+1}$. Moreover, $\binom{rp^\alpha+\beta}{j}$ ($r = 1, \ldots, p-1$) is divisible by p^γ and not by $p^{\gamma+1}$.*

The number $N = \binom{rp^\alpha+\beta}{j}$ may be written

$$N = \frac{(rp^\alpha+\beta)(rp^\alpha+\beta-1)\ldots(rp^\alpha+1)rp^\alpha(rp^\alpha-1)\ldots(rp^\alpha-p)\ldots(rp^\alpha-2p)\ldots(rp^\alpha-np)\ldots}{1\cdot 2\ldots(p-1)p(p+1)\ldots(2p-1)2p(2p+1)\ldots(np-1)np(np+1)\ldots(np+\beta+1)\ldots} = \frac{A}{B}.$$

In B the multiples of p occur in periods of p. The first $\beta+1$ factors in A contain the factor p^α. Let the rest be regarded as forming cycles of period p and make the kth cycle in the numerator correspond to the kth in B. Then the highest power of p in the factor kp in B is canceled by that in $rp^\alpha - kp$ in the corresponding cycle in A and $\gamma = \alpha$. Any one of a number of exceptions can occur if the last cycle in B is incomplete with respect to the above correspondence so that some factors p must be canceled by factors of p^α from the first $\beta+1$ factors in A. Then $\gamma = \alpha - s < \alpha$. It is not necessary to trace these exceptions in greater detail for, whatever exceptional case

presents itself, it is certain that it will be repeated, and result in the same value of s in the last cycle of the denominator of

$$N' = \frac{(rp^{\alpha-\delta}-\beta)(rp^{\alpha-\delta}-\beta-1)\ldots(rp^{\alpha-\delta}-1)rp^{\alpha-\delta}\ldots(rp^{\alpha-\delta}-p)\ldots(rp^{\alpha-\delta}-2p)\ldots(rp^{\alpha-\delta}-np)\ldots}{1\cdot 2\ldots(p-1)p(p+1)\ldots(2p-1)2p(2p+1)\ldots(np-1)np(np+1)\ldots(np+\beta+1)\ldots}$$

Thus, in all cases, N' is divisible by $p^{\gamma+\delta}$ ($\gamma = \alpha - s$) and not by $p^{\gamma+\delta+1}$.

THEOREM. *If the highest power of p which divides N is p^{γ}, then the highest power of p which divides $N_1 = \binom{rp^{\alpha}+\beta}{j+sp^{\alpha}}$ is p^{γ}.* ($r = 1, \ldots, p-1$; $s = 0, \ldots, r$; $j = 0, \ldots, rp^{\alpha}+\beta-sp^{\alpha}$; $\alpha = 1, 2, \ldots, \infty$).

We have $N = p^{\gamma} R$.

$$N_1 = p^{\gamma} R \frac{(rp^{\alpha}-\beta-j)(rp^{\alpha}-\beta-j-1)\ldots(rp^{\alpha}-ap)(rp^{\alpha}-ap-1)\ldots(rp^{\alpha}-a-1p)\ldots(rp^{\alpha}-\beta-j-sp^{\alpha}+1)}{(j+1)(j+2)\ldots(j+ap+2+\beta-j)(\beta+ap+3)(\beta+ap+4)\ldots(j+sp^{\alpha})}$$

Inasmuch as the number of factors in the numerator and denominator of the multiplier of $p^{\gamma} R$ in N_1 is sp^{α}, the first factor in this numerator which is divisible by p, viz., $rp^{\alpha}-ap$, is of such a form that $j+(t-\beta) = ap = a_1 p^{\sigma}$, where a_1 is prime to p, and, since $t < p$, we have $t-\beta < p$. Hence the power p^{σ} in the denominator factor $j+(t-\beta)$ cancels the highest power of p, viz., p^{σ}, in the corresponding factor $rp^{\alpha}-ap$ in the numerator, and the same comment applies in the case of the other factors $rp^{\alpha}-(a+1)p$, $rp^{\alpha}-(a+2)p, \ldots$, which are divisible by a power of p. Thus the multiplier of $p^{\gamma} R$ in N_1 is incongruent to zero modulo p, which proves the theorem.

The plexus Q of positive integers, each less than the prime p, obtained by reducing the numbers of Pascal's triangle according to the modulus p possesses properties of symmetry that are partly exhibited by the arrangement of the numbers of Q in geometric forms. Supposing Q to be written as an equilateral triangle of indefinite length with 1, 1 as the first row, 1, 2, 1 as the second row and so on, there is lateral symmetry with reference to the median and various periodic recurrences, vertical as well as lateral.

In row number p the end elements are 1 and the other $p-1$ are zeros forming the base of an inverted equilateral triangle of zeros with vertex on the median. These triangles recur in horizontal rows, two in the second row with bases in row number $2p$, three in the third row with bases in row number $3p$, and so on to infinity. In general, row number p^r is a row of p^r-1 zeros (end elements 1) the base of an inverted equilateral triangle of $p^r(p^r-1)/2$ zero elements forming the first of a series of rows of triangles of these dimensions, two triangles in the second row with bases in row

number $2p^r$ of the plexus, three in the third row with bases in row number $3p^r$, and so on to infinity.

It follows from the two theorems of the present paragraph that, within an arbitrarily chosen triangle of zeros in the cycle of rows from row number p^r of Q, to row number p^{r+1}, there are triangles of dimensions $p-1$ that are submerged to the depth r, that is, their elements are also elements of zero triangles brought forward from all previous cycles. We shall say that the depth of submergence w of a chosen zero element of Q is the order of that zero (0^w). There results the following:

THEOREM. *If w is the order of a chosen zero element in the plexus of residues of Pascal's triangle reduced modulo p, a prime, then the corresponding combinatory number is divisible by p^w and not by p^{w+1}.*

By means of formulas involving parameters of restricted numerical variation* we can delimit the triangle of general dimensions and general position whose elements are all zeros of order w and thus obtain a determination of all combinatory numbers which are divisible by p^w and not by p^{w+1}. For this purpose we write the following formulas each one of which represents a set of combinatory numbers belonging to the totality found in the binomial expansions from $(a+b)^{p^r}$ to $(a+b)^{p^{r+1}}$, or, as we shall say, belonging to the cycle (p^r, p^{r+1}):

$$\binom{(p^{r-1}+t_1)p+j}{i+j}.$$

(I)

$$(r=1,\ldots\infty;\ t_1=0,\ldots,p^{r-1}(p-1)-1;\ j=0,\ldots,p-2;$$
$$i=(s-1)p+1,\ldots,sp-1-j;\ s=1,\ldots,p^{r-1}+t_1),$$

$$\binom{(p^{r-2}+t_2)p^2+j}{i+j}$$

(II)

$$(r=1,\ldots\infty;\ t_2=0,\ldots,p^{r-2}(p-1)-1;\ j=0,\ldots,p^2-2;$$
$$i=(s-1)p^2+1,\ldots sp^2-1-j;\ s=1,\ldots,p^{r-2}+t_2),$$

$$\binom{(p^{r-3}+t_3)p^3+j}{i+j}$$

(III)

$$(r=1,\ldots\infty;\ t_3=0,\ldots,p^{r-3}(p-1)-1;\ j=0,\ldots,p^3-2;$$
$$i=(s-1)p^3+1,\ldots sp^3-1-j;\ s=1,\ldots,p^{r-3}+t_3),$$

* Trans. Amer. Math. Soc., vol. 20 (1919), p. 155.

. . . . . .

$$(R) \qquad \binom{(1+t_r)p^r+j}{i+j}$$

$$(r = 0, \ldots, \varkappa;\ t_r = 0, \ldots, p-2;\ j = 0, \ldots, p^r-2;$$
$$i = (s-1)p^r+1, \ldots, sp^r-1-j;\ s = 1, \ldots, 1+t_r).$$

The formula (I) represents numbers which are in triangles brought forward from the first cycle, formula (II) numbers in triangles from the second cycle, and so on. The previous results establish the following theorem.

THEOREM. *Combinatory numbers of the r-th cycle* (p^r, p^{r+1}) *which are divisible by* p^w *and not by* p^{w+1} *are precisely and only those which are found in* w *and only in* w *of the sets* (I), ..., (*R*).

Thus the desired numbers are obtained by forming the greatest common subset of each group of w sets selected from (I), (II), ..., (*R*), and combining these subsets. For example, we find as the complete set of numbers of the second cycle $(3^2, 3^3)$, which are divisible by 3^2 and not by 3^3, the following:

$$\binom{9}{1}\ \binom{9}{2}\ \binom{10}{2}\ \binom{9}{4}\ \binom{9}{5}\ \binom{10}{5}\ \binom{9}{7}\ \binom{9}{8}\ \binom{10}{8}\ \binom{12}{4}\ \binom{12}{5}\ \binom{13}{5}$$

$$\binom{12}{7}\ \binom{12}{8}\ \binom{13}{8}\ \binom{15}{7}\ \binom{15}{8}\ \binom{16}{8}\ \binom{18}{1}\ \binom{18}{2}\ \binom{19}{2}\ \binom{18}{4}\ \binom{18}{5}\ \binom{19}{5}$$

$$\binom{18}{7}\ \binom{18}{8}\ \binom{19}{8}\ \binom{18}{10}\ \binom{18}{11}\ \binom{19}{11}\ \binom{18}{13}\ \binom{18}{14}\ \binom{19}{14}\ \binom{18}{16}\ \binom{18}{17}\ \binom{19}{17}$$

$$\binom{21}{4}\ \binom{21}{5}\ \binom{22}{5}\ \binom{21}{7}\ \binom{21}{8}\ \binom{22}{8}\ \binom{21}{13}\ \binom{22}{14}\ \binom{21}{14}\ \binom{21}{16}\ \binom{21}{17}\ \binom{22}{17}$$

$$\binom{24}{7}\ \binom{24}{8}\ \binom{25}{8}\ \binom{24}{16}\ \binom{25}{17}\ \binom{24}{17}.$$

7. **The period parallelogram.** The additive sequence number $2x$ of the plexus R of paragraph 5 is congruent modulo p to $\varrho S(2x)$ where

$$S(2x):\ 4^r(x+1),\ 4^r(x+2),\ \ldots,\ 4^r(p-1),\ 0,\ 4^r(p+1),\ \ldots.$$

Hence, with $p > 2$, since by Fermat's theorem $2^{p-1} \equiv 1 \pmod{p}$, the $(p-1)$-th additive sequence is congruent to the initial series $S_1(\alpha)$, so spaced that it

begins with the term which is congruent to $[\frac{1}{2}(p-1)+1]\varrho$ and has its first zero on the first drift line. Therefore, with reference to the modulus p, not only the original sequence $S_1(\alpha)$ but also all additive sequences in their turn repeat throughout the plexus R in cycles of p. It is also evident that the configuration of residues contained between any two consecutive drift lines is identical with that between any other consecutive pair. Hence the entire plexus is a net formed by parallelograms of identical content the bounds of each being a pair of consecutive drift lines and two consecutive recurrences of the initial series $S_1(\alpha)$. This parallelogram, here and in the general theory to follow, is designated as the period parallelogram of the plexus of figurate numbers. If we delimit by formulas the period parallelogram of general position within the plexus, we have a complete analytical description, or, as we may say, a definitive residual evaluation of all numbers of the infinite plexus.

The initial number α_i on the nth drift line is $\alpha_{np} = \varrho np \equiv 0 \pmod{p}$, and that of the $(n+1)$-th drift line is $\alpha_{(n+1)p} = (n+1)\varrho p \equiv 0 \pmod{p}$. The initial series $S_1(\alpha)$ repeats as column number $1+m(p-1)$ and next again as column number $1+(m+1)(p-1)$. Hence the four corner elements of the period parallelogram of general position in R are the following:

$$D\colon \begin{pmatrix} m(p-1)+1 \\ \frac{1}{2}m(p-1)+(n+1)p \end{pmatrix}. \qquad C\colon \begin{pmatrix} (m+1)(p-1)+1 \\ \frac{1}{2}(m+1)(p-1)+(n+1)p \end{pmatrix},$$

$$A\colon \begin{pmatrix} m(p-1)+1 \\ \frac{1}{2}m(p-1)+np \end{pmatrix}. \qquad B\colon \begin{pmatrix} (m+1)(p-1)+1 \\ \frac{1}{2}(m+1)(p-1)+np \end{pmatrix}.$$

These formulas may be written explicitly be means of (1_1). Thus

$$\begin{pmatrix} m(p-1)+1 \\ \frac{1}{2}m(p-1)+np \end{pmatrix} = \binom{\frac{1}{2}m(p-1)}{0}(np-\tfrac{1}{2}m(p-1))\varrho$$

$$+\binom{\frac{1}{2}m(p-1)}{1}(np-\tfrac{1}{2}m(p-1)+1)\varrho+\cdots$$

$$+\binom{\frac{1}{2}m(p-1)}{\frac{1}{2}m(p-1)}(np+\tfrac{1}{2}m(p-1))\varrho \pmod{p}.$$

A particular case is added showing the entire period parallelogram for $p=5$, $\varrho=4$ (cf. fig. 1).

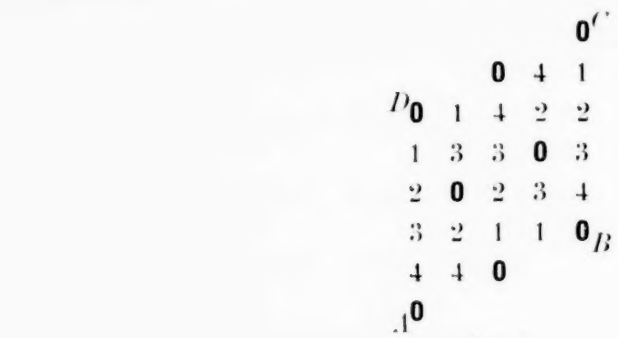

Fig. 3.

The definitive residual evaluation of this case of the plexus R is given, therefore, by the following 22 formulas:

$$\left(\frac{4m+1}{2m+5n}\right) \equiv 0, \quad \left(\frac{4m+1}{2m+5n+1}\right) \equiv 4, \quad \left(\frac{4m+1}{2m+5n+2}\right) \equiv 3 \pmod 5,$$

$$\left(\frac{4m+1}{2m+5n+3}\right) \equiv 2, \quad \left(\frac{4m+1}{2m+5n+4}\right) \equiv 1, \quad \left(\frac{4m+1}{2m+5n+5}\right) \equiv 0 \pmod 5,$$

$$\left(\frac{4m+2}{2m+5n+1}\right) \equiv 4, \quad \left(\frac{4m+2}{2m+5n+2}\right) \equiv 2, \quad \left(\frac{4m+2}{2m+5n+3}\right) \equiv 0 \pmod 5,$$

$$\left(\frac{4m+2}{2m+5n+4}\right) \equiv 3, \quad \left(\frac{4m+2}{2m+5n+5}\right) \equiv 1, \quad \left(\frac{4m+3}{2m+5n+1}\right) \equiv 0 \pmod 5,$$

$$\left(\frac{4m+3}{2m+5n+2}\right) \equiv 1, \quad \left(\frac{4m+3}{2m+5n+3}\right) \equiv 2, \quad \left(\frac{4m+3}{2m+5n+4}\right) \equiv 3 \pmod 5,$$

$$\left(\frac{4m+3}{2m+5n+5}\right) \equiv 4, \quad \left(\frac{4m+3}{2m+5n+6}\right) \equiv 0, \quad \left(\frac{4m+4}{2m+5n+2}\right) \equiv 1 \pmod 5,$$

$$\left(\frac{4m+4}{2m+5n+3}\right) \equiv 3, \quad \left(\frac{4m+4}{2m+5n+4}\right) \equiv 0, \quad \left(\frac{4m+4}{2m+5n+5}\right) \equiv 2 \pmod 5,$$

$$\left(\frac{4m+4}{2m+5n+6}\right) \equiv 4 \pmod 5,$$

where $m = 0, 1, \ldots \infty$; $n = 1, 2, \ldots \infty$; and $\alpha_i \equiv i\varrho \pmod p$ as in $S_1(\alpha)$, paragraph 5.

8. **General theory.** The elements of $S(\alpha)$ which are included between two consecutive drift lines of a skew-symmetrical plexus P form a permutation.

involving repetitions, of the p least residues modulo p, and the same is true of $S(\alpha')$, $S(\alpha'')$, ... Moreover, each element is a polynomial in the quantities $\alpha_1, \alpha_2, \ldots$ of the type

$$(4_1) \qquad \alpha = c_1 \alpha_{r+1} + c_2 \alpha_{r+2} + \cdots + c_s \alpha_{r+s}, \qquad (r \text{ a fixed integer}),$$

where the coefficients c_i are residues of p. Since the number of different sets $(c_1, c_2, \ldots, c_s)$ is finite, we have the following theorem.

THEOREM. *Every skew-symmetrical plexus P_1 is congruent modulo p, a prime, to a net of period parallelograms of finite dimensions and identical content.*

To make a more explicit determination of the width of this parallelogram it is necessary to deal with particular series $S(\alpha)$, as was done above in the case of $S_1(\alpha)$. If we assume, for $S(\alpha)$, any sequence whose residues are the cyclic permutations of the p residues repeated in the order

$$(4) \quad (0, 1, 2, \ldots, p-1), (1, 2, \ldots, p-1, 0), (2, 3, \ldots p-1, 0, 1), \ldots,$$

then the width of the period parallelogram is $p(p-1)$.

The null elements upon drift lines do not constitute the totality of zeros in P. The rest are arranged, in general, in polygons forming a set of geometric figures which recur with the recurrence of the period parallelogram. The reason for this symmetry is involved with the combination of the symmetry inherent in the drift line theory with that of the null element configurations in Pascal's triangle reduced modulo p (cf. §6). It is difficult to obtain a general analytical description of these null element polygons but in a particular case (cf. fig. 2) they present a beautifully variegated symmetry.

The period parallelogram of the plexus of figure 2, which is an example of the general case (4), is subjoined.

						0 C
				0	**0**	2
D		**0**	1	**0**	2	**0**
0	2	1	2	2	1	2
2	2	1	**0**	2	1	1
0	2	2	2	2	**0**	2
2	**0**	**0**	**0**	1	2	1
1	**0**	**0**	1	1	2	**0**
2	**0**	1	**0**	1	1	1
1	1	2	1	**0**	**0**	**0** B
0	1	2	2	**0**		
1	1	**0**				
0 A						

Fig. 4.

The six null elements, (mod. 3), forming the enclosed triangle of zeros are

$$\left(\frac{6n-4}{9m+3n-9}\right), \qquad \left(\frac{6n-4}{9m+3n-8}\right),$$

$$\left(\frac{6n-4}{9m+3n-7}\right), \qquad \left(\frac{6n-3}{9m+3n-8}\right),$$

$$\left(\frac{6n-3}{9m+3n-7}\right), \qquad \left(\frac{6n-2}{9m+3n-7}\right),$$

$$(m, n = 1, \ldots \infty).$$

THE UNIVERSITY OF PENNSYLVANIA.

ON SYMMETRIC FORMS IN N VARIABLES. II.

By Arnold Dresden.

1. In a previous paper with the same title, published in these Annals, it was shown that a symmetric form of given weight S, given number of parts n and given distribution of weight over the parts, $\left(\prod_{l=1}^{n} f^{s_l - s_{l+1}}\right)$, where $s_1 \geq s_2 \geq \cdots \geq s_n$, $s_{n+1} = 0$ and $\sum_{i=1}^{n} s_i = S$, can be expressed in the form $\sum' \lambda(r_1, \ldots, r_n; s_1, \ldots, s_n) \prod_{l=1}^{n} (1^{r_l})$. In this formula, the sum is to be extended over all the non-decreasing partitions $r_1, \ldots, r_n$ of S, such that $\sum_{1}^{i} r_l \geq \sum_{1}^{i} s_l$ for $i < n$. This relation between two equipartite partitions (r) and (s) is called the dominance relation, and is represented by the formula $(r)\,D(s)$. A particular multiplier $\lambda(r_1, \ldots, r_n; s_1, \ldots, s_n)$ is determined by means of the formulae

$$\lambda(s_1, \ldots, s_n; s_1, \ldots, s_n) = 1,$$

$$\sum \lambda(t_1, \ldots, t_n; s_1, \ldots, s_n) \sum \prod_{i,j=1}^{n,i} \binom{r_{ij} - r_{i,j-1}}{k_{ij}} = 0, \tag{1}$$

In this formula the outer sum is to be extended over all the partitions (t) of S, such that $(r)\,D(t)\,D(s)$; the inner sum is extended over all possible solutions in non-negative integers k_{ij} of the system of linear equations

$$\sum_{j=1}^{i} k_{ij} = t_i, \qquad \sum_{j=i}^{n} k_{ji} = r_i. \tag{2}$$

As observed in the paper referred to above, the direct determination of the multipliers λ becomes quite laborious when the number of parts of the form increases. It is the object of the present paper to develop some general theorems which will facilitate their calculation. Doubtless still further simplifications can be made, at least theoretically. It appears however that further simplifications in the methods of calculation are accompanied by an increase in the labor involved in applying them to concrete cases.

2. We begin by effecting an arrangement of the partitions (r) which satisfy the condition $(r)\,D(s)$, of such a character that none of them is dominated by one which precedes it. For this purpose, we designate by V_j the substitution which replaces the elements r_j and r_{j+1} of any partition (r) by r_j+1 and $r_{j+1}-1$ respectively. It is then evident that we may write $V_j[(s)]\,D(s)$. The arrangement referred to is now obtained in the following manner: Write (s), $V_{n-1}[(s)]$, $V_{n-1}^2[(s)]$, ... $V_{n-1}^{\alpha_1}[(s)]$, where $\alpha_1 = s_{n-2}-s_{n-1}$; next we write $V_{n-2}[(s)]$ and apply to it the substitution V_{n-1} as often as will still yield a non-increasing partition, repeat the same process with V_{n-2}^2, ... $V_{n-2}^{\alpha_2}$, where $\alpha_2 = s_{n-3}-s_{n-2}$; we now write $V_{n-3}[(s)]$ and apply to it the substitutions V_{n-1} and V_{n-2} as often as is possible and in the order in which they were applied to (s), then proceeding with iterations of V_{n-3}. In this manner we proceed, applying to any partition $V_{n-i}[(s)]$ all the possible combinations of the substitutions V_{n-1}, V_{n-2}, ... V_{n-i-1} and their iterations in the order in which they were applied to (s), until we reach the partition $S, 0, \ldots, 0$, which dominates all the others. While it is not so that in this arrangement of the partitions which dominate (s), any one will dominate all those which precede it, we can affirm that none is dominated by any partition which precedes it.* A sequence of partitions arranged in accordance with the above plan will be called a *normally ordered* sequence of partitions.

If, for example, (s) be the four-partite partition of 9 given by 4221, the normally ordered sequence of partitions (r) for which $(r)\,D(s)$ will be as follows:

(s) : 4221;	$V_1V_2V_3[(s)]$: 5220;	$V_1^2V_2^3V_3[(s)]$: 6300;
$V_2[(s)]$: 4311;	$V_1V_2^2V_3[(s)]$: 5310;	$V_1^3V_2^2V_3[(s)]$: 7110;
$V_2V_3[(s)]$: 4320;	$V_1V_2^3V_3[(s)]$: 5400;	$V_1^3V_2^3V_3[(s)]$: 7200;
$V_2^2V_3[(s)]$: 4410;	$V_1^2V_2[(s)]$: 6111;	$V_1^4V_2^3V_3[(s)]$: 8100;
$V_1V_2[(s)]$: 5211;	$V_1^2V_2^2V_3[(s)]$: 6210;	$V_1^5V_2^3V_3[(s)]$: 9000.

In accordance with the theorem proved in the earlier paper the symmetric form (1²34), i. e., $\sum x_1^4x_2^3x_3x_4$, is expressible in terms of products of elementary symmetric functions $E_{i_1}E_{i_2}E_{i_3}E_{i_4}$ in which $i_1i_2i_3i_4$ represents any one of the partitions of 9 listed above. To determine the multipliers $\lambda(i_1i_2i_3i_4;4321)$ with which these products $E_{i_1}E_{i_2}E_{i_3}E_{i_4}$ appear in the

* This statement will not seem strange when we realize that the relation D is not connected in the field of partitions of S, i. e., that if (u) and (r) are two partitions of S, then it may happen that neither $(u)\,D(r)$, nor $(r)\,D(u)$.

final formula, we use formula (1), taking for $t_1 \cdots t_n$ all those partitions in the above list which precede $i_1\, i_2\, i_3\, i_4$ and which it dominates. Since we know that $\lambda(4221; 4221) = 1$, we have thus obtained a systematic method for calculating all the multipliers.

3. If equation (1) is to be used for determining a particular multiplier $\lambda(r_1, \ldots, r_n; s_1, \ldots, s_n)$, this one will appear in the equation with the coefficient 1; we will say that it is the "leading term" in the equation. The coefficient of $\lambda(t_1, \ldots, t_n; s_1, \ldots, s_n)$ in the equation in which $\lambda(r_1, \ldots, r_n; s_1, \ldots, s_n)$ is leading term will be denoted by $C[(r); (t)]$. We know then that $C[(r); (r)] = 1$ and that $C[(r); (t)] = \sum \prod_{i,j=1}^{n,i} \binom{r_{ij} - r_{i,j+1}}{k_{ij}}$, the sum being extended over all non-negative integral solutions k_{ij} of the equations $\sum_{j=1}^{i} k_{ij} = t_i$, $\sum_{j}^{n} k_{ji} = r_i$, while $r_{ij} = \sum_{l=j}^{i} k_{lj}$.

We establish now the following theorems:

THEOREM I. *Equations* (1) *written successively for the partitions of a normally ordered sequence of partitions uniquely determine the multipliers* $\lambda((r); (s))$ *for all those partitions* (4) *for which* $(r)\, D(s)$.

In order to determine $\lambda((r); (s))$, we write the equation (1) in which $\lambda((r); (s))$ is the leading term. This equation will then involve terms in $\lambda((t); (s))$ for partitions (t) for which $(r)\, D(t)\, D(s)$, i. e., for partitions (t) which precede (r) in the normally ordered sequence. Since we know moreover that $\lambda((s); (s)) = 1$, the truth of the theorem is apparent.

THEOREM II. *If* V_l^{-1} *denotes the substitution inverse to* V_l *and if* $e_l = r_l - r_{l-1}$ $C[(r); V_l^{-1}[(r)]] = e_l$ *and* $\lambda((r); V_l^{-1}[(r)]) = -e_l$.

In equations (2) we have for this case $t_j = r_j$, except for $j = l$ and $l+1$, while $t_l = r_l - 1$ and $t_{l+1} = r_{l+1} + 1$. It follows that $k_{11} = r_1, \ldots, k_{l-1,l-1} = r_{l-1}$, $k_{ll} = r_l - 1$, $k_{l-1,l+1} = r_{l+1}, \ldots, k_{nn} = r_n$; $k_{l+1,l} = 1$, while every other $k_{ij} = 0$, constitutes the only admissible solution of equations (2). Consequently every factor in $C[(r); V_l^{-1}[(r)]]$ reduces to 1, except $\binom{r_{l-1,l} - r_{l+1,l-1}}{k_{l-1,l}}$; moreover $r_{l-1,l} = k_{ll} + k_{l-1,l} = r_l$ and $r_{l-1,l+1} = r_{l+1}$. Hence $C[(r); V_l^{-1}[(r)]] = e_l$. To prove the second part of the theorem, we observe that the only partitions (t) for which $(r)\, D(t)\, DV_l^{-1}[(r)]$ are (r) and $V_l^{-1}[(r)]$ themselves. Therefore equation (1) reduces in this case to the simple form

$$C[(r); V_l^{-1}[(r)]]\ \lambda(V_l^{-1}[(r)]; V_l^{-1}[(r)]) + C[(r); (r)]\ \lambda((r); V_l^{-1}[(r)]) = 0.$$

In view of the preceding result, we may therefore conclude that $\lambda((r); V_l^{-1}[(r)]) = -e_l$.

COROLLARY. $\lambda(V_l[(r)];(r)) = -(e_l+2)$.

This theorem is a special case of

THEOREM III. *With the notations introduced in Theorem* II, *we have* $C[(r);\ V_l^{-j}[(r)]] = \binom{e_l}{j}$ *and* $\lambda((r);\ V_l^{-j}[(r)]) = \frac{1}{j}(-1)^j e_l \binom{e_l-j-1}{j-1}$.

Equations (2) again have a unique solution in this case, viz., $k_{11} = r_1, \ldots, k_{l-1,l-1} = r_{l-1}, k_{ll} = r_l - j, k_{l+1,l+1} = r_{l+1}, \ldots, k_{nn} = r_n$; $k_{l-1,l} = j$ and every other $k_{ij} = 0$. As in Theorem II this solution reduces every factor of the coefficient C to unity, except the factor $\binom{r_{l-1,l}-r_{l-1,l+1}}{k_{l-1,l}}$, so that we find $C[(r);\ V_l^{-j}[(r)]] = \binom{e_l}{j}$. In order to prove the second part of the theorem, we establish the following

LEMMA: $\binom{a+2p}{p} + \sum_{l=1}^{p} \frac{(-1)^l}{l}(a+2l)\binom{a+l-1}{l-1}\binom{a+2p}{p-l} = 0$.*

Denoting the sum of the first m terms on the left by s_m, we verify readily that

$$s_1 = \binom{a+2p}{p} - (a+2)\binom{a+2p}{p-1} = -\frac{a+2p}{p}\binom{a+1}{1}\binom{a+2p-1}{p-2},$$

and that $s_2 = \frac{a+2p}{p}\binom{a+2}{2}\binom{a+2p-1}{p-3}$. Assuming now that for $m < p$, we have $s_m = \frac{(-1)^m(a+2p)}{p}\binom{a+m}{m}\binom{a+2p-1}{p-m-1}$, which is true for $m = 1, 2$, it follows that

$$s_{m+1} = s_m + \frac{(-1)^{m+1}}{m+1}\binom{a+m}{m}\binom{a+2p}{p-m-1}(a+2m+2)$$

$$= \frac{(-1)^{m+1}(a+2p)}{p}\binom{a+m+1}{m+1}\binom{a+2p-1}{p-m-2}.$$

The formula assumed for s_m is therefore true for $m = 1, \ldots, p-1$. For the latter value of m, it yields the result, that $s_{p-1} = \frac{(-1)^{p-1}(a+2p)}{p}\binom{a+p-1}{p-1}$, from which it follows that $s_p = 0$.

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* This formula is a special case of the following more general one: $\binom{a+2p-r}{p} + \sum_{l=r+1}^{p} \frac{(-1)^{l-r}}{l}(a+2l-r)\binom{a+l}{r}\binom{a+l-r-1}{l-r-1}\binom{a+2p-r}{p-l} = 0$, which may be proved by induction starting form the formula in the lemma, to which this one reduces for $r = 0$.

The partitions (t) for which $(r)\,D(t)\,DV_l^{-j}[(r)]$, are clearly (r), $V_l^{-1}[(r)]$, ... $V_l^{-j}[(r)]$; hence equation (1) now takes the following form:

$$\sum_{i=0}^{j} C[(r);\ V_l^{-i}[(r)]]\ \lambda(V_l^{-i}[(r)];\ V_l^{-j}[(r)]) = 0.$$

This equation, in virtue of Theorem I, determines $\lambda((r); V_l^{-j}[(r)])$ uniquely. And now it is seen that this equation is satisfied by the formula for $\lambda((r); V_l^{-j}[(r)])$ stated in the theorem. For, if we write $(r') = V_l^{-i}[(r)]$, then we would have $\lambda(V_l^{-i}[(r)];\ V_l^{-j}[(r)]) = \lambda((r');\ V_l^{-(j-i)}[(r')]) = \dfrac{(-1)^{j-i}}{j-i}\, e_l' \binom{e_l'-j+i-1}{j-i-1}$; moreover $e_l' = r_l' - r_{l-1}' = r_l - i - (r_{l-1}+i) = e_l - 2i$. Hence, if we make use of the formula proved in the first part of this theorem, it remains to show, that $\sum_{i=0}^{j-1} \dfrac{(-1)^{j-i}}{j-i}\,(e_l-2i)\binom{e_l}{i}\binom{e_l-j-i-1}{j-i-1} + \binom{e_l}{j} = 0$. But this is what the formula of our lemma reduces to when we replace a, l and p by $e_l - 2j$, $j-i$ and j respectively.

COROLLARY. $\lambda(V_l^j[(r)];(r)) = \dfrac{1}{j}(-1)^j\,(e_l+2j)\binom{e_l+j-1}{j-1}$.

THEOREM IV. *If $\alpha_1, \ldots, \alpha_{n-1}$ be an arbitrary set of non-negative integers and $\alpha_m = 0$ $(m < n-1)$, then* $C[V_1^{\alpha_1}\cdots V_{n-1}^{\alpha_{n-1}}[(r)];(r)] = C[V_1^{\alpha_1}\cdots V_{m-1}^{\alpha_{m-1}}[(r)];(r)]\cdot C[V_{m+1}^{\alpha_{m+1}}\cdots V_{n-1}^{\alpha_{n-1}}[(r)];(r)]$.

If (r') designate the partition $V_1^{\alpha_1}\cdots V_{n-1}^{\alpha_{n-1}}[(r)]$, we have

$$r_1' = r_1+\alpha_1,\quad r_2' = r_2-\alpha_1+\alpha_2,\quad r_3' = r_3-\alpha_2+\alpha_3,\ \ldots,$$

$$r_{n-1}' = r_{n-1}-\alpha_{n-2}+\alpha_{n-1},\quad r_n' = r_n-\alpha_{n-1}.$$

Hence, since $\alpha_m = 0$, the partition (r') may be split into two parts, $r_1', \ldots, r_m'$ and $r_{m+1}', \ldots, r_n'$, such that $\sum_{i=1}^{m} r_i' = \sum_{i=1}^{m} r_i$ and consequently $\sum_{i=m+1}^{n} r_i' = \sum_{i=m+1}^{n} r_i$. The equations upon the solution of which the determination of $C[(r');(r)]$ depends are those of the system

$$\sum_{j=1}^{i} k_{ij} = r_i \qquad \sum_{j=i}^{n} k_{ji} = r_i' \qquad (i = 1, 2, \ldots, n). \tag{3}$$

We consider now also the systems of equations

$$\sum_{j=i}^{i} k_{ij} = r_i, \ (i = 1, 2, \ldots, n); \quad \sum_{j=i}^{n} k_{ji} = r'_i, \ (i = 1, 2, \ldots, m);$$
$$\sum_{j=i}^{n} k_{ji} = r_i, \ (i = m+1, \ldots, n) \tag{4}$$

and

$$\sum_{j=1}^{i} k_{ij} = r_i, \ (i = 1, \ldots, n); \quad \sum_{j=i}^{n} k_{ji} = r_i, \ (i = 1, \ldots, m);$$
$$\sum_{j=i}^{n} k_{ji} = r'_i, \ (i = m+1, \ldots, n), \tag{5}$$

which determine the coefficients $C[V_1^{\sigma_1} \cdots V_{m-1}^{\sigma_{m-1}}[(r)]; (r)]$ and $C[V_{m-1}^{\sigma_{m+1}} \cdots \cdots V_{n-1}^{\sigma_{n-1}}[(r)]; (r)]$ respectively. Using now the same procedure for the solution of these equations as was used in the proofs of Theorems II and III, we find that, if the set k_{ij} constitute a solution of equations (3), and if we put

$$k'_{ij} = k_{ij}, \text{ for } i \leq m, \ j = 1, \ldots, i; \ k'_{ij} = 0, \text{ for } i > m, \ j \neq i; \ k'_{ii} = r_i, \text{ for } i > m,$$

then the set k'_{ij} furnishes a solution for the system (4); and if we put

$$k''_{ij} = k_{ij}, \text{ for } j > m, \ i = j, \ldots, n; \ k''_{ij} = 0, \text{ for } j \leq m, \ i \neq j; \ k''_{jj} = r_j, \text{ for } j \leq m,$$

the set k''_{ij} gives a solution of equations (5). On the other hand, if the sets k'_{ij} and k''_{ij} are solutions of equations (4) and (5) respectively and if we put $k_{ij} = k'_{ij}$ for $i \leq m$, $k_{ij} = k''_{ij}$ for $i > m$, then the set k_{ij} is a solution of the system (3). We conclude from this that any term in $C[(r'); (r)]$ is the product of a term in $C[V_1^{\sigma_1} \cdots V_{m-1}^{\sigma_{m-1}}[(r)]; (r)]$ by a term in $C[V_{m-1}^{\sigma_{m+1}} \cdots \cdots V_{n-1}^{\sigma_{n-1}}[(r)]; (r)]$ and that, conversely, any such product is a term of $C[(r'); (r)]$. Our theorem is therefore proved.

THEOREM V. *The coefficient* $C[(r); V_l^{-1} \cdots V_{l-k}^{-1}[(r)]]$ *is equal to*

$$\sum_{i=1}^{k-1} \Sigma \prod (j_m - j_n) e_{j_1} e_{j_2} \cdots e_{j_i}.$$

In this formula, the inner sum is to be extended over all *increasing* sets of indices, i at a time, selected from $l, l+1, \ldots, l+k$; the product is to be

taken over all the *positive* differences of adjacent indices, and the coefficient of terms consisting of a single factor e_i is to be taken equal to 1.

For $k = 0$, the theorem asserts that $C[(r); V_l^{-1}[(r)]] = e_l$, which was proved to be the case in Theorem II. It remains to show that the formula holds for $k+1$ provided it does for k, and we find it convenient to introduce for this purpose the abbreviation C_{lk} for $C[(r); V_l^{-1} \cdots V_{l+k}^{-1}[(r)]]$.

The determination of C_{lk} depends upon the solution of equations (2), in which $t_i = r_i$, except for $i = l$ or $l+k+1$, while $t_l = r_l - 1$ and $t_{l+k+1} = r_{l+k+1} + 1$. It follows then that we must have $k_{ij} = 0$ for $j < l$, $i \neq j$ and also for $i > l+k+1$, $j \neq i$, and $k_{ii} = r_i$ for $i < 1$ and for $i > l+k+1$. Consequently the solution of equations (2) reduces in this case to the solution of the reduced system

$$(6) \qquad \sum_{j=l}^{i} k_{ij} = t_i, \qquad \sum_{j=i}^{l+k+1} k_{ji} = r_i, \qquad l \leqq i \leqq l+k+1,$$

and the coefficient C_{lk} depends on the reduced system alone, because the further equations of system (2) yield only unit factors for the coefficient. If we make use now of the special values of $t_l, \ldots, t_{l+k+1}$ which were specified above, it follows that $k_{ll} = r_l - 1$ and that, since $\sum_{i=l}^{n} k_{il} = r_l$, one and only one of the indices $k_{l+1,l}, \ldots, k_{l+k+1,l}$ must be equal to 1, all the others being equal to 0. If we take $k_{l+1,l} = 1$, then there remains to be solved the system

$$\sum_{j=l+1}^{i} k_{ij} = t_i, \qquad \sum_{j=i}^{l+k+1} k_{ji} = r_i, \qquad l+1 \leqq i \leqq l+k+1,$$

which is exactly the system (6), after l and k have been replaced by $l+1$ and $k-1$ respectively and which therefore determines $C_{l+1,k-1}$. Moreover, since $k_{ll} = r_l - 1$, $k_{l+1,l} = 1$ and consequently $k_{l+1,l+1} = r_{l+1} - 1$, we obtain for C_{lk} the factor $e_l + 1$. We find therefore for C_{lk} the term $(e_l+1)C_{l+1,k-1}$.

If we take $k_{l+1,l} = k_{l+2,l} = \cdots = k_{l+\alpha,l} = k_{l+\alpha+2,l} = \cdots = k_{n,l} = 0$, $k_{l+\alpha+1,l} = 1$, $(0 < \alpha < k)$, then we must have $k_{ij} = 0$ for $j = l+1, \ldots, l+\alpha$, $i \neq j$, and $k_{ii} = r_i$ for $i = l+1, \ldots, l+\alpha$. There remains then to be solved the system

$$\sum_{j=l+\alpha+1}^{i} k_{ij} = t_i, \qquad \sum_{j=i}^{l+k+1} k_{ji} = r_i, \qquad l+\alpha+1 \leqq i \leqq l+k+1,$$

which is obtainable from the system of equations (6) by replacing l and k by $l+\alpha+1$ and $k-\alpha-1$ respectively and which therefore gives rise to $C_{l-\alpha+1,k-\alpha-1}$. But, since $k_{l-\alpha-1,l} = 1$ we obtain the further factor e_l, so that we have found for C_{lk} the term $e_l C_{l+\alpha+1,k-\alpha-1}$.

It remains to consider the solution of (6) in which we have $k_{l+1,l} = \cdots = k_{l-k,l} = 0$, $k_{l+k+1,l} = 1$. This solution gives for C_{lk} the single factor e_l. We conclude therefore that

$$C_{lk} = (e_l+1)C_{l+1,k-1} + e_l\left[\sum_{\alpha=1}^{k-1} C_{l+\alpha+1,k-\alpha-1}+1\right].$$

For $k+1$, this becomes:

$$C_{l,k+1} = (e_l+1)C_{l+1,k} + e_l\left[\sum_{\alpha=1}^{k} C_{l+\alpha+1,k-\alpha}+1\right]. \tag{7}$$

Assuming now the truth of the formula, asserted in the theorem, for $1, 2, \ldots, k$, it follows that $C_{l+1,k}$ will contain all the products of $1, 2, \ldots, k+1$ factors which can be formed from $e_{l+1}, \ldots, e_{l+k+1}$; that $C_{l+2,k-1}$ will contain all the products of $1, 2, \ldots, k$ factors which can be formed from $e_{l+2}, \ldots, e_{l+k+1}$; $\ldots$; that $C_{l+\alpha+1,k-\alpha}$ will contain all the products of $1, 2, \ldots, k-\alpha+1$ factors which can be formed from $e_{l+\alpha+1}, \ldots, e_{l+k+1}$; $\ldots$. Consequently, on the right hand side of (7) will occur every product of $1, 2, \ldots, k+2$ factors that can be formed from $e_l, \ldots, e_{l+k+1}$. For, the products that do not contain e_l as a factor will appear in the term $C_{l+1,k}$; and the products which do contain e_l as a factor will consist of e_l multiplied by a product of factors selected from $e_{l+1}, \ldots, e_{l+k+1}$, or from $e_{l+2}, \ldots, e_{l+k+1}, \ldots$, or from $e_{l+\alpha+1}, \ldots, e_{l+k+1}, \ldots$ and will therefore occur in $e_l C_{l+1,k}$, or in $e_l C_{l+2,k-1}, \ldots$, or in $e_l C_{l+\alpha+1,k-\alpha}, \ldots$ respectively; and the single factor e_l also occurs on the right of (7).

Moreover, those terms which do not contain the factor e_l will appear in $C_{l,k+1}$ with the same coefficient that they had in $C_{l+1,k}$, i. e., with the coefficient demanded by the theorem. Consider now a term in $C_{l,k+1}$ which does have the factor e_l. Let its factor with next lowest index be e_{l+j}; then it will occur in each of the products $e_l C_{l+1,k}$, $e_l C_{l+2,k-1}, \ldots, e_l C_{l+j,k-j-1}$ and in each of these products with the same coefficient, viz., the product of all positive differences of adjacent indices from $l+j$ on, occurring in it. Consequently its coefficient in $C_{l,k+1}$ will have the additional factor j, which is indeed the one new positive difference between adjacent indices which appears in the term. Our theorem has therefore been proved.

THEOREM VI. *If* $R = \sum_{i=1}^{n} r_i$ *and if* (R) *designates the partition of* R *which consists of the parts* $R, 0, \ldots, 0$, *then we have the formula* $C[(R); (r)] = \prod_{i=1}^{n} \binom{r_1 + r_2 + \cdots + r_i}{r_i}$.

For this case, equations (2) reduce to the following form:

$$\sum_{j=1}^{i} k_{ij} = r_i; \qquad \sum_{j=1}^{n} k_{j1} = R, \qquad \sum_{j=i}^{n} k_{ji} = 0 \text{ for } i = 2, \ldots, n.$$

The last $n-1$ of these equations give $k_{ij} = 0$ for $i = 2, \ldots, n$, $j = 1, \ldots, i$. Substituting these values in the first n equations, we obtain $k_{i1} = r_i$. This leads further to $r_{i1} = \sum_{j=1}^{i} r_j$, $r_{ij} = 0$ for $j > 1$ and hence to the formula stated in the theorem.

THEOREM VII. *If the set* k_{ij} *be any solution of equations* (2) *and if a system of equations* (2′) *be set up in which* $t_i' = t_i$ *and* $r_i' = r_i$, *except that* $r_l' = r_l + 1$ *and* $r_{l-1}' = r_{l-1} - 1$, *then all the solutions of* (2′) *may be obtained from those of* (2) *by applying to each of the latter each of the following substitutions:*

(8) *replace* k_{il} *and* $k_{i,l-1}$ *by* $k_{il} + 1$ *and* $k_{i,l-1} - 1$ *respectively, for* $l < i \leq n$,

(9) *replace* k_{ll}, $k_{l\beta}$, $k_{\alpha\beta}$ *and* $k_{\alpha,l-1}$ *by* $k_{ll} + 1$, $k_{l\beta} - 1$, $k_{\alpha\beta} + 1$ *and* $k_{\alpha,l+1} - 1$ *respectively, for* $l < \alpha \leq n$, $1 \leq \beta < l$.

It is apparent that any set k_{ij}', obtained by use of any of the substitutions mentioned in the theorem will satisfy equations (2′), provided the set k_{ij} satisfy equations (2). Conversely, if k_{ij}' be a set which satisfies (2′), then either (*a*) $k_{ll}' < r_l + 1$ and at least one of the $k_{il}' > 0$ $(l < i \leq n)$; or (*b*) $k_{ll}' = r_l + 1$ and $k_{il}' = 0$ $(l < i \leq n)$. In case (*a*), suppose that $k_{jl}' > 0$; then we define a set k_{ij} by putting $k_{jl} = k_{jl}' - 1$ and $k_{j,l-1} = k_{j,l+1}' - 1$ and every other k_{ij} equal to the corresponding k_{ij}'. This set k_{ij} is then evidently a solution of the system (2), from which the given solution of (2′) is obtained by means of one of the substitutions (8) mentioned in the theorem. In case (*b*), we observe that, since $t_l = \sum_{j=1}^{l} k_{lj}'$, we must have $t_l > r_l$. On the other hand, $\sum_{i=1}^{l} t_i \leq \sum_{i=1}^{l} r_i$, because (*r*) $D(t)$; therefore $\sum_{i=1}^{l-1} r_i - \sum_{i=1}^{l-1} t_i > 0$, so that not all the k_{ij}' for which $i < l$ and $j \geq l$ can be equal to zero. But, further than this, not all the k_{ij}' for which $j < l$ and $i > l$ can vanish; for, if they did, we would have $\sum_{i=1}^{l-1} r_i - \sum_{i=1}^{l-1} t_i = k_{11}' + \cdots + k_{l,l-1}' = t_l - k_{ll}' = t_l - r_l - 1$,

and hence $\sum_{i=1}^{l} r_i - \sum_{i}^{l} t_i = -1$, which contradicts the fact that $(r) \, D \, (t)$. We can therefore find at least one $k'_{\alpha\beta}$, which is positive, and such that $l < \alpha$ and $l > \beta$. Now we define a set k_{ij} by putting $k_{ii} = k'_{ii} - 1 = r_i$, $k_{\alpha,l-1} = k'_{\alpha,l-1} + 1$, $k_{\alpha\beta} = k'_{\alpha\beta} - 1$, $k_{l\beta} = k'_{l\beta} + 1$ and every other k_{ij} equal to the corresponding k'_{ij}. This set k_{ij} is then manifestly a solution of equations (2), from which the given solution of (2′) is obtained by means of one of the substitutions (9). This proves the theorem.

COROLLARY 1. *All the sets k'_{ij} which determine $C[(r); V_l[(t)]]$ can be obtained from the sets k_{ij} which determine $C[(r); (t)]$ by applying to each of the latter each of the following substitutions:*

replace k_{li} and $k_{l-1,i}$ by $k_{li}+1$ and $k_{l-1,i}-1$ respectively, for $1 \leq i < l$,

replace $k_{l-1,l-1}$, $k_{\alpha,l-1}$, $k_{\alpha\beta}$ and $k_{l\beta}$ by $k_{l-1,l-1}-1$, $k_{\alpha,l-1}+1$, $k_{\alpha\beta}-1$ and $k_{l\beta}+1$ respectively, for $l+1 < \alpha \leq n$, $1 \leq \beta < l+1$.

COROLLARY 2. *If (r) is the partition $V_j \cdots V_{j+k}[(t)]$ and $j \leq l \leq j+k$, then all the sets k'_{ij} which determine $C[V_l[(r)]; (t)]$ can be obtained from the sets k_{ij} which determine $C[(r); (t)]$ by applying to each of the latter each of the substitutions (8).*

In this special case of Theorem VII, case (b) can not arise, because we have now $r_l = t_l$ or $r_l = t_l + 1$, so that k'_{ll} can not be equal to $r_l + 1$.

COROLLARY 3. *If (r) represents the partition $V_j \cdots V_{j+k}[(t)]$ and $j \leq l < j+k$, all the sets k'_{ij} which determine $C[(r); V_l[(t)]]$ can be obtained from the sets k_{ij} which determine $C[(r); (t)]$ by applying to each of the latter each of the following substitutions:*

replace k_{li} and $k_{l-1,i}$ by $k_{li}+1$ and $k_{l-1,i}-1$ respectively, for $1 \leq i \leq l$.

4. The theorems developed in the preceding section will now be used for the calculation of a particular symmetric form, whose determination in terms of elementary functions by the ordinary methods and also by the methods used in our earlier paper would be a matter of a great deal of labor. We shall take the example already brought forward in 2, where the normal sequence of partitions for this example was set up. We have seen that $\sum x_1^4 x_2^3 x_3 x_4 = \sum \lambda (i_1 i_2 i_3 i_4 ; 4221) E_{i_1} E_{i_2} E_{i_3} E_{i_4}$ the sum being extended over all the partitions (i) of the normal sequence given above. We turn now to the determination of the multipliers $\lambda (i_1 i_2 i_3 i_4 ; 4221)$.

(a) We know that $\lambda (4221 ; 4221) = 1$ and from Theorem II, Corollary, we obtain the result that $\lambda (4311 ; 4221) = -2$.

(b) For $V_2 V_3 [(s)]$, equation (1) takes the form

$$\lambda (4320 ; 4221) + C[4320 ; 4311] \lambda (4311 ; 4221) + C[4320 ; 4221] = 0.$$

$C[4320 ; 4311] = 2$, by means of Theorem II; $C[4320 ; 4221] = 1 + 2 + 1 \cdot 2 = 5$ by means of Theorem V. Hence we find $\lambda (4320 ; 4221) = -1$.

(c) We shall determine $\lambda(4410; 4221)$ by use of equations (1) and (2). Equation (1) becomes

$$\lambda(4410; 4221) + C[4410; 4320]\,\lambda(4320; 4221) + C[4410; 4311]\,\lambda(4311; 4221) + C[4410; 4221] = 0.$$

From Theorem II follows that $C[4410; 4320] = 3$; from Theorem V follows that $C[4410; 4311] = 7$.

To determine $C[4410; 4221]$, making use of equations (2), we have to determine the k_{ij}. To do this we write equations (2) in the following symbolic form, it being understood that the sum of the variables in any row (column) must be equal to the number written in front of (below) it:

$$\begin{array}{llllll} 4 & k_{11} & & & & \\ 2 & k_{21} & k_{22} & & & \\ 2 & k_{31} & k_{32} & k_{33} & & \\ 1 & k_{41} & k_{42} & k_{43} & k_{44} & \\ \hline & 4 & 4 & 1 & 0. & \end{array}$$

Of these equations we find two solutions, viz.:

$$\begin{array}{llll} 4 & & & \\ 0 & 2 & & \\ 0 & 1 & 1 & \\ 0 & 1 & 0 & 0 \end{array} \quad \text{and} \quad \begin{array}{llll} 4 & & & \\ 0 & 2 & & \\ 0 & 2 & 0 & \\ 0 & 0 & 1 & 0, \end{array}$$

in which the values of the unknowns are written in place of the unknowns themselves in the symbolic form of the system of equations. This way of writing the systems k_{ij} will prove to be well adapted to the application of Theorem VII and its Corollaries. Arranging the values of r in a similar triangular array, we find for this case the following result:

$$\begin{array}{llll} 4 & & & \\ 4 & 2 & & \\ 4 & 3 & 1 & \\ 4 & 4 & 1 & 0 \end{array} \quad \text{and} \quad \begin{array}{llll} 4 & & & \\ 4 & 2 & & \\ 4 & 4 & 0 & \\ 4 & 4 & 1 & 0; \end{array}$$

from this we obtain $C[4410; 4221] = 2 \cdot 3 + \binom{4}{2} \cdot 1 = 12$.

We conclude that $\lambda(4410; 4221) = 5$.

(*d*) For $\lambda(5211; 4221)$, equation (1) reduces to

$$\lambda(5211; 4221) + C[5211; 4311]\ \lambda(4311; 4221) + C[5211; 4221] = 0,$$

because 5221 does not dominate any of the partitions which precede it in the normal sequence except 4311 and 4221. By means of Theorems II and V, we find from the above equation that $\lambda(5211; 4221) = -1$.

(*e*) The coefficients in equation (1) for the determination of $\lambda(5220; 4221)$ are found as follows:

$C[5220; 5211] = 2$ and $C[5220; 4320] = 3$, by means of Theorem II.

$C[5220; 4311] = C[5211; 4311] \times C[4320; 4311] = 6$, by means of Theorems IV and II.

$C[5220; 4221] = 3 + 0 + 2 + 2 \cdot 3 \cdot 2 = 17$ by use of Theorem V.

We conclude that $\lambda(5220; 4221) = 0$.

(*f*) For the determination of the coefficients in the equation for $\lambda(5310; 4221)$, the same methods suffice except for $C[5310; 4221]$. Since $5310 = V_1[4410]$, the sets k'_{ij} which determine the latter coefficient may be found by means of Theorem VII from the sets k_{ij} which determine $C[4410; 4221]$. To these sets, exhibited in (*e*), we apply substitution (8) for $l = 1$ and for $i = 2, 3, 4$; we find in this way the following sets:

4	4	4	4	4
1 1	0 2	0 2	1 1	0 2
0 1 1	1 0 1	0 1 1	0 2 0	1 1 0
0 1 0 0,	0 1 0 0,	1 0 0 0,	0 0 1 0,	0 0 1 0,

from which we derive the result $C[5310; 4221] = 8 + 6 + 4 + 12 + 6 = 36$ and $\lambda(5310; 4221) = 5$.

(*g*) In order to determine $\lambda(5400; 4221)$ we begin by evaluating $C[5400; 4221]$. Since $5400 = V_2[5310]$, the sets k'_{ij} for this coefficient can be obtained from those found in (*f*) by use of Theorem VII for $l = 2$ and $i = 3, 4$. This gives the following results:

(10)

4	4	4
1 1	0 2	0 2
0 2 0	1 1 0	0 2 0
0 1 0 0,	0 1 0 0,	1 0 0 0,

from which we derive $C[5400; 4221] = 78$.

Since $4311 = V_2[4221]$, we shall find the sets for $C[5400; 4311]$ by applying Corollary 1 of Theorem VII for $l = 2$ and $i = 1, 2$ to the sets (10); in this way we find

$$(11)\qquad \begin{array}{lll} 4 & 4 & 4 \\ 1\ 2 & 0\ 3 & 0\ 3 \\ 0\ 1\ 0 & 1\ 0\ 0 & 0\ 1\ 0 \\ 0\ 1\ 0\ 0, & 0\ 1\ 0\ 0, & 1\ 0\ 0\ 0, \end{array}$$

from which follows $C[5400; 4311] = 48$.

Using the same Corollary again, we find from sets (11) those for $C[5400; 4320]$ and $C[5400; 5211]$, viz.,

$$(12a)\qquad \begin{array}{lll} 4 & 4 & \\ 1\ 2 & 0\ 3 & \\ 0\ 2\ 0 & 1\ 1\ 0 & \\ 0\ 0\ 0\ 0, & 0\ 0\ 0\ 0, & \end{array} \qquad (12b)\qquad \begin{array}{l} 5 \\ 0\ 2 \\ 0\ 1\ 0 \\ 0\ 1\ 0\ 0 \end{array}$$

respectively, from which we derive $C[5400; 4320] = 22$ and $C[5400; 5211] = 12$.

To determine $C[5400; 4410]$ we may apply Theorem V or else apply Corollary 1 of Theorem VII once more, this time to the sets (12a) with $l = 2$ and $i = 1, 2$; in either case we find $C[5400; 4410] = 9$.

For $C[5400; 5220]$ we apply Theorem III with $l = j = 2$, yielding the result $C[5400; 5220] = \binom{4}{2} = 6$. Finally, we have from Theorem II, $C[5400; 5211] = 4$. Combining these results we obtain $\lambda(5400; 4221) = -13$.

(*b*) The processes applied above suffice for the rapid determination of the remaining multipliers and illustrate sufficiently the way in which the theorems developed in 3 are used. We find thus:

$$\lambda(6111; 4221) = 5, \qquad \lambda(6210; 4221) = -13, \quad \lambda(6300; 4221) = 9,$$

$$\lambda(7110; 4221) = -11, \quad \lambda(7200; 4221) = 20, \qquad \lambda(8100; 4221) = 11.$$

(*j*) The coefficients in the equation for $\lambda(9000; 4221)$ are most readily found by means of Theorem VI, because the formula there proved obviates the necessity of determining the sets k_{ij}. We obtain in this way the following result:

$$C[9000;\,4221] = \binom{6}{2}\binom{8}{2}\binom{9}{1} = 3780, \quad C[9000;\,4311] = \binom{7}{3}\binom{8}{1}\binom{9}{1} = 2520,$$

$$C[9000;\,4320] = \binom{7}{3}\binom{9}{2} = 1260. \qquad C[9000;\,4410] = \binom{8}{4}\binom{9}{1} = 630, \text{ etc.}$$

The calculations carried out in (a)—(i) enable us to express the symmetric form $\sum x_1^4 x_2^3 x_3 x_4$ in terms of the elementary symmetric functions E_i as follows:

$$\sum x_1^4 x_2^3 x_3 x_4 = E_4 E_2^2 E_1 - 2 E_4 E_3 E_1^2 - E_4 E_3 E_2 + 5 E_4^2 E_1 -$$

$$E_5 E_2 E_1^2 + 5 E_5 E_3 E_1 - 13 E_5 E_4 + 5 E_6 E_1^3 - 13 E_6 E_2 E_1 + 9 E_6 E_3$$

$$- 11 E_7 E_1^2 + 20 E_7 E_2 + 11 E_8 E_1 - 27 E_9.$$

The University of Wisconsin,
September 1922.

ON AN INFINITE SYSTEM OF NON-ABELIAN GROUPS OF ORDER nm^{n-1}.*

By W. E. Edington.

Consider two non-commutative operators s_1 and s_2, both of order n, which also satisfy the condition that $s_1^\alpha s_2^\alpha$ and $s_1^\beta s_2^\beta$ are commutative for $\alpha, \beta = 1, 2, 3, \ldots, (n-1)$. Before determining the necessary and sufficient conditions for the definition and existence of finite groups, certain general relations growing out of the commutativity of $s_1^\alpha s_2^\alpha$ and $s_1^\beta s_2^\beta$ will be established.

Lemma I. *If* $s_1^n = s_2^n = 1$, $s_1 s_2 \neq s_2 s_1$, *and if* $s_1^\alpha s_2^\alpha$ *and* $s_1^\beta s_2^\beta$ *are commutative for* $\alpha, \beta = 1, 2, \ldots, (n-1)$, *then* $s_1^\alpha s_2^\alpha$ *and* $s_2^\beta s_1^\beta$ *are commutative.*

This follows readily from the fact that $s_2^\beta s_1^\beta = (s_1^{-\beta} s_2^{-\beta})^{-1}$ and that, if $s_1^\alpha s_2^\alpha$ is commutative with $s_1^{-\beta} s_2^{-\beta}$, it is commutative with the inverse of $s_1^{-\beta} s_2^{-\beta}$.

Lemma II. *If* $s_1^n = s_2^n = 1$, $s_1 s_2 \neq s_2 s_1$, *and if* $s_1^\alpha s_2^\alpha$ *and* $s_1^\beta s_2^\beta$ *are commutative for* $\alpha, \beta = 1, 2, \ldots, (n-1)$, *then*

$$(1) \qquad [s_1^\alpha s_2^{\alpha+\beta}]^\delta = s_1^{\alpha-(\delta-1)\beta} s_2^{\alpha-(\delta-2)\beta} (s_1^{\alpha+\beta} s_2^{\alpha+2\beta})^{\delta-1},$$

$$(2) \qquad [s_1^{\alpha+\gamma\beta} s_2^{\alpha+(\gamma+1)\beta}]^n = [s_1^{\alpha+(\gamma+1)\beta} s_2^{\alpha+(\gamma-2)\beta}]^n,$$

$$(3) \qquad [s_1^\alpha s_2^{\alpha+\beta}]^\delta = s_1^\alpha s_2^\alpha \cdot s_1^{\alpha-\beta} s_2^{\alpha-\beta} \cdot \ldots$$

$$\ldots \cdot s_1^{\alpha-(\delta-1)\beta} s_2^{\alpha-(\delta-1)\beta} \cdot s_2^\beta s_1^\beta \cdot s_2^{2\beta} s_1^{2\beta} \cdot \ldots \cdot s_2^{(\delta-1)\beta} s_1^{(\delta-1)\beta} \cdot s_2^{\delta\beta},$$

where $\delta = 0, 1, 2, \ldots,$ *and* $\gamma = 0, 1, 2, \ldots, (n-1)$.

The proofs of these relations are by mathematical induction. Assume (1) to hold. Then

$$[s_1^\alpha s_2^{\alpha+\beta}]^{\delta+1} = s_1^\alpha \cdot s_2^{\alpha+\beta} s_1^{\alpha+\beta} \cdot s_1^{-\delta\beta} s_2^{-\delta\beta} \cdot s_2^{\alpha+2\beta} (s_1^{\alpha+\beta} s_2^{\alpha+2\beta})^{\delta-1}$$

$$= s_1^{\alpha-\delta\beta} s_2^{\alpha-(\delta-1)\beta} (s_1^{\alpha+\beta} s_2^{\alpha+2\beta})^\delta.$$

* Presented to the American Mathematical Society, Dec. 29, 1922.

Hence the relation (1) holds when δ is replaced by $\delta+1$, and it also holds for $\delta=1$ and $\delta=2$, for

$$(s_1^\alpha s_2^{\alpha+\beta})^2 = s_1^{\alpha-\beta}\cdot s_1^\beta s_2^\beta \cdot s_2^\alpha s_1^\alpha \cdot s_2^{\alpha+\beta} = s_1^{\alpha-\beta} s_2^\alpha \cdot s_1^{\alpha+\beta} s_2^{\alpha+2\beta},$$

which establishes the relation.

Now set $\delta=n$ and (1) reduces to

$$(s_1^\alpha s_2^{\alpha+\beta})^n = (s_1^{\alpha+\beta} s_2^{\alpha+2\beta})^n,$$

which is relation (2) for $\gamma=0$. The proof of (2) follows in a manner similar to that used for (1).

To prove relation (3), we have

$$(s_1^\alpha s_2^{\alpha+\beta})^\delta = s_1^\alpha s_2^\alpha \cdot s_2^\beta \cdot (s_1^\alpha s_2^{\alpha+\beta})^{\delta-1} = s_1^\alpha s_2^\alpha \cdot s_2^\beta s_1^\beta \cdot s_1^{\alpha-\beta} s_2^{\alpha-\beta} \cdot s_2^{2\beta} \cdot (s_1^\alpha s_2^{\alpha+\beta})^{\delta-2},$$

from which, by continuing the process, (3) is obtained. The proof is easily completed by induction. The proof may also be gotten by a reduction of the relation (1) to the relation (3).

Setting $\gamma=0, 1, 2, \ldots, (n-2)$, successively in (2), we have

$$(4) \qquad (s_1^\alpha s_2^{\alpha+\beta})^n = (s_1^{\alpha+\beta} s_2^{\alpha+2\beta})^n = \ldots = (s_1^{\alpha-\beta} s_2^\alpha)^n.$$

When $\beta=\alpha$, this reduces to

$$(5) \qquad (s_1^\alpha s_2^{2\alpha})^n = (s_1^{2\alpha} s_2^{3\alpha})^n = \ldots = (s_2^\alpha)^n = 1.$$

Hence, when α is prime to n, s_2^α is of order n and the orders of the products $s_1^{\mu\alpha} s_2^{(\mu+1)\alpha}$, $\mu=0, 1, 2, \ldots, (n-1)$, are n and divisors of n. If n is a prime number, all the products of the form $s_1^\tau s_2^\varphi$, $\tau \neq \varphi$, are of order n, for α then has $n-1$ values prime to n and μ takes values from 1 to $n-2$ to give products of the form under discussion, and therefore (5) gives $(n-1)(n-2)$ distinct products of this form which is the greatest number possible. If n is not prime, the number of values that α may take that are prime to n is $\varphi(n)$ and, keeping in mind the values that μ may take, it follows that the greatest number of operators of the form $s_1^\tau s_2^\varphi$, $\tau \neq \varphi$, of order n then given by (5) is $(n-2)\varphi(n)$.

It is evident on account of the commutativity of the operators $s_1^\alpha s_2^\alpha$, $s_1^{\alpha+\beta} s_2^{\alpha+\beta}, \ldots, s_2^\beta s_1^\beta, \ldots, s_2^{(\delta-1)\beta} s_2^{(\delta-1)\beta}$, that in some developments of (3) an operator and its inverse will appear and hence reduce to unity. In what follows it will be assumed that all such operators and their inverses which appear in (3) have been reduced to unity. $s_2^{\delta\beta}$ will be of order n when $\delta\beta$ is prime to n; otherwise its order will be equal to the quotient of n divided by the greatest common divisor of n and $\delta\beta$, or, in other words, it is equal to the least common multiple of $\delta\beta$ and n divided by $\delta\beta$. The order of $(s_1^\alpha s_2^{\alpha+\beta})^\delta$ is the greatest when $\delta\beta$ and n are relatively prime and then its order is equal to, or is a divisor of, the least common multiple of the orders of the operators of the type $s_1^\varphi s_2^\varphi$ and $s_2^\varphi s_1^\varphi$, and n. When $\delta\beta$ is not relatively prime to n the order of $(s_1^\alpha s_2^{\alpha+\beta})^\delta$ is equal to or is a divisor of the least common multiple of the orders of the operators of the form $s_1^\varphi s_2^\varphi$ and $s_2^\varphi s_1^\varphi$ and the least common multiple of $\delta\beta$ and n divided by $\delta\beta$. The order of $s_1^\alpha s_2^{\alpha+\beta}$ can be found by choosing for δ the least value for which the congruence $\delta\beta \equiv 0 \bmod n$ is satisfied. The least common multiple of this value of δ and the orders of the operators of the form $s_1^\varphi s_2^\varphi$ and $s_2^\varphi s_1^\varphi$ will be the highest order $s_1^\alpha s_2^{\alpha+\beta}$ can have. That the order may be less than this for special values of α and β depending on n follows as a result of the following:

LEMMA III. *If* $s_1^n = s_2^n = 1$, $s_1 s_2 \neq s_2 s_1$, *and if* $s_1^\alpha s_2^\alpha$ *and* $s_1^\beta s_2^\beta$ *are commutative for* $\alpha, \beta = 1, 2, \ldots, (n-1)$, *then*

$$(6) \quad (s_1^\alpha s_2^{\alpha+\beta})^\delta = s_1^{\alpha-(\delta-1)\beta} s_2^{\alpha-(\delta-2)\beta} s_1^{\alpha-(\delta-2)\beta} \cdots s_2^{\alpha-2\beta} s_1^{\alpha-\beta} s_2^{\alpha+\delta\beta},$$

and, if a value of $\varkappa$ *exists for which* $\varkappa\beta - \alpha \equiv 0 \bmod n$, *where* $\delta\beta \equiv \varkappa\beta - \alpha \equiv 0 \bmod n$, *then the order of* $s_1^\alpha s_2^{\alpha+\beta}$ *is that value of* δ *for which* $\delta\beta \equiv 0 \bmod n$.

The proof of (6) follows readily from the relation (1) by applying the relation (1) to $(s_1^{\alpha+\beta} s_2^{\alpha+2\beta})^{\delta-1}$ in the right member of (1) and continuing this application of (1) to the powers in the right member of (1) $\delta-1$ times successively. The second part of the above lemma is true; for, when the congruence can be satisfied, the values of δ and $\varkappa$ for which it is satisfied are such that the right member of (6) reduces to unity, since one of the powers of s_1 or s_2 in the right member of (6) reduces to unity thus bringing about the products of inverses in succession. This relation (6) will determine orders of operators of the type $s_1^\tau s_2^\varphi$, $\tau \neq \varphi$, whose orders are not dependent on the orders of the operators of the form $s_1^\varphi s_2^\varphi$. Hence the order of all operators of the form $s_1^\alpha s_2^{\alpha+\beta}$, where the congruence $\varkappa\beta \equiv \alpha \bmod n$ can be satisfied, is n or a divisor of n.

The results of the preceding lemmas are summarized in the following:

THEOREM I. *If s_1 and s_2 are two non-commutative operators which fulfil the conditions*

$$s_1^n = s_2^n = 1, \qquad s_1^\alpha s_2^\alpha s_1^\beta s_2^\beta = s_1^\beta s_2^\beta s_1^\alpha s_2^\alpha,$$

$$\alpha, \beta = 1, 2, \ldots, (n-1),$$

and if the orders of all operators of the type $s_1^\alpha s_2^\alpha$ are known, then the maximum orders of all operators of the type $s_1^\tau s_2^\varphi$, $\tau \neq \varphi$, are known.

The commutative operators $s_1^\alpha s_2^\alpha$, $\alpha = 1, 2, \ldots, (n-1)$, generate an abelian group whose order is at most the least common multiple of the orders of its generators. This group is invariant under s_1 and s_2; for

$$s_1^{-1} s_1^\alpha s_2^\alpha s_1 = s_1^{\alpha-1} s_2^{\alpha-1} \cdot s_2 s_1 = s_1^{\alpha-1} s_2^{\alpha-1} (s_1^{-1} s_2^{-1})^{-1},$$

and

$$s_2^{-1} s_1^\alpha s_2^\alpha s_2 = s_2^{-1} s_1^{-1} \cdot s_1^{\alpha+1} s_2^{\alpha+1} = (s_1 s_2)^{-1} s_1^{\alpha+1} s_2^{\alpha+1},$$

which are in the abelian group. No power of either s_1 or s_2 is in this abelian group without both s_1 and s_2 to that power being in it; for, if s_1^φ is in this abelian group, then $s_1^\varphi \cdot s_1^{-\varphi} s_2^{-\varphi} = s_2^{-\varphi}$ is in it and hence s_2^φ is also in it. It will now be assumed that no power of s_1 or s_2, except the identity is in this abelian group. Then the order of the group generated is n times the least common multiple of the orders of the operators $s_1^\alpha s_2^\alpha$, $\alpha = 1, 2, \ldots, (n-1)$.

Let now s_1 and s_2 fulfil the following conditions:

$$s_1^n = s_2^n = (s_1^\alpha s_2^\alpha)^m = 1, \quad s_1^\alpha s_2^\alpha s_1^\beta s_2^\beta = s_1^\beta s_2^\beta s_1^\alpha s_2^\alpha, \quad \alpha, \beta = 1, 2, \ldots, (n-1).$$

Then the order of the group generated is $n m^{n-1}$. Such a group exists for every value of m and n, for let

$$s_1 = abcd \ldots efgh \cdot ijkl \ldots mnop \cdot \ldots \cdot qrst \ldots uvwx,$$

$$s_2 = ihgfe \ldots dcb \cdot axwvu \ldots tsr \cdot q \ldots \ldots \ldots - ponm \ldots lkj,$$

where s_1 and s_2 are each composed of m cycles of order n.

Then

$$s_1 s_2 = ai \ldots q \cdot hx \ldots p,$$

$$s_1^2 s_2^2 = ai \ldots q \cdot gw \ldots o,$$

$$s_1^3 s_2^3 = ai \ldots q \cdot fv \ldots n,$$

$$\cdots\cdots\cdots\cdots$$

$$s_1^{n-1} s_2^{n-1} = ai \ldots q \cdot bx \ldots j.$$

Hence the operators $s_1^\alpha s_2^\alpha$ are all of order m and by observation it is evident that they are all commutative with each other. By adding another cycle of order n to s_1, and constructing s_2 anew the generality of these generators is proved. s_1 and s_2 then are each composed of $m+1$ cycles of order n and may be represented as

$$s_1' = abcd \ldots efgh \cdot ijkl \ldots mnop \cdot \ldots\ldots \cdot qrst \ldots uvwx \cdot \alpha\beta\gamma\delta \ldots \varepsilon\theta\mu q,$$

$$s_2' = ihgfe \ldots dcb \cdot aq\mu\theta\varepsilon \ldots \delta\gamma\beta \cdot \alpha xwvu \ldots tsr \cdot q \ldots \cdot -ponm \ldots lkj,$$

and the products of equal powers of s_1' and s_2' will all be found to be of order $m+1$. Hence, if these generators hold for m cycles of order n they hold for $m+1$ cycles. It may be easily verified that they hold for $m = 2, 3, 4$ cycles and hence they hold for any number of cycles by mathematical induction. When $n = 2$, the groups generated are the dihedral groups represented as regular groups. When $n = 3$, $m = 2$, the group G is the tetrahedral group, and for $n = 3$, $m = 3$, the group is the non-abelian G_{27} all of whose operators are of order 3. $n = 4$, $m = 2$, gives the G_{32} which is also defined by the conditions $s_1^4 = s_2^4 = (s_1 s_2)^2 = (s_1^2 s_2^2)^2 = 1$. Taking $n = 3$ the conditions become $s_1^3 = s_2^3 = (s_1 s_2)^m = 1$, $s_1 s_2 s_1^2 s_2^2 = s_1^2 s_2^2 s_1 s_2$. From this last condition it follows immediately that $(s_1^2 s_2)^3 = 1$. In the author's thesis* it was proved that $t_1^3 = t_2^3 = (t_1 t_2)^3 = (t_1 t_2^2)^\alpha = 1$ defined a group of order $3\alpha^2$, and this is the same group as the one defined by $s_1^3 = s_2^3 = (s_1 s_2)^m = (s_1 s_2^2)^3 = 1$, since the last definition may be obtained from the first by setting $s_2 = t_2^2$, $s_1 = t_1$, and $\alpha = m$. Hence for $n = 3$ the group is of order $3m^2$.

The general group G of order $n m^{n-1}$ defined above contains an invariant abelian group of order m^{n-1} all of whose operators are of order m or

* Trans. Amer. Math. Soc., vol. 25, 1923.

divisors of order m. Call this abelian subgroup H. G is generated by s_1 and H. If $\varkappa$ is a divisor of n, then the group generated by $s_1^{\varkappa}$ and H is invariant under G, for, if this group contains $s_1^{\varkappa}$, it will also contain $s_2^{\varkappa}$ and hence be invariant under s_2 since

$$s_2^{-1} s_1^{\varkappa} s_1^{\alpha} s_2^{\alpha} s_2 = s_2^{-1} s_1^{\varkappa+\alpha} s_2^{\alpha+1} = s_2^{-1} s_1^{-1} s_1^{\varkappa+\alpha+1} s_2^{\varkappa+\alpha+1} s_2^{-\varkappa},$$

which is in the group generated by $s_1^{\varkappa}$ and H; for $s_1^{\alpha} s_2^{\alpha}$ is any generator of H. Therefore the group G is solvable.

All the above is summarized in

THEOREM II. *If two non-commutative operators s_1 and s_2 fulfil the conditions*

$$s_1^n = s_2^n = (s_1^{\alpha} s_2^{\alpha})^m = 1, \quad s_1^{\alpha} s_2^{\alpha} s_1^{\beta} s_2^{\beta} = s_1^{\beta} s_2^{\beta} s_1^{\alpha} s_2^{\alpha}, \quad \alpha\beta = 1, 2, \ldots, (n-1),$$

the group of greatest order generated by s_1 and s_2 is of order nm^{n-1}, and such a group exists for every value of m and n. This group is a solvable group.

PURDUE UNIVERSITY,
December, 1921.

A CONTRIBUTION TO THE THEORY OF CLOSED CHAINS.

By A. Arwin.

In a previous paper* I have established a theory of periodically closed chains which takes a place between the general theory of the domains of numbers and Gauss's classical theory of forms. In the present paper I intend to develop this theory along several lines.

In § 1 I define a new symbol $\left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\}$, that may possibly be called an "ideal fraction" because of its connection with the symbol of the ideal numbers $(Q_\rho, P_\rho + \sqrt{D})$ in $K(\sqrt{D})$. Then I prove the fundamental theorem: Two equivalent fractions, which are multiplied by each other, i. e., which are compounded according to Gauss's rule for composition of forms, give equivalent fractions. In the theory of forms it is said: The composition of equivalent forms produces equivalent forms. As a consequence of the uniquely stated "multiplication" we get the unique decomposition of a fraction into simple fractions, "prime fractions". In § 2 I shall prove the following theorem, which was mentioned in the previous paper: If $D \equiv 1\ (8)$, $D \equiv 5\ (8)$ and the equation $M^2 - DN^2 = \pm 4$ is soluble, the numbers of the closed chains in the two types are equal. If $D \equiv 2, 3\ (4)$ there exists only the first type. If $D \equiv 5\ (8)$ and $M^2 - DN^2 \neq \pm 4$, the number in the first type is thrice the number of the second type. We also find that the number of the ideal classes is equal to the closed chains of the second type when existing (if $D \equiv 2, 3\ (4)$ it is equal simply to that of the only existing first type), and only in the case of $D \equiv 5\ (8)$ and $M^2 - DN^2 \neq \pm 4$ is the number of the closed chains, which always equals the number of Gauss's forms, the quadruple of the ideal classes. In § 3 the law of quadratic reciprocity will be proved in an elementary, new way in virtue of our theory of closed chains. In § 4 I shall treat the symmetrical chains which hold only fractions of the second category, that is, the case of $D \equiv 1\ (4)$ and $N(\eta) = M^2 - DN^2 = +1$, and I shall prove that the solution of $M^2 - DN^2 = +4$, when existing, and always in the case of $M^2 - DN^2 = +1$, must have the special form

$$(y_\chi^2 + z_\chi^2 + y_{\chi-1}^2 + z_{\chi-1}^2)^2 - (2^\varepsilon b)^2 D = 4 \qquad (\varepsilon = 0 \text{ or } 1)$$

* Periodically closed chains of reduced fractions, Ann. of Math., vol. 24, p. 39.

where $y_\chi, z_\chi, y_{\chi-1}, z_{\chi-1}$ satisfy $y_\chi z_{\chi-1} - y_{\chi-1} z_\chi = \pm 1$, and the continued fraction of y_χ / z_χ will give the equivalence

$$\left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\} = \left\{\frac{P_\tau + \sqrt{D}}{Q_\tau}\right\} \qquad D = P_\rho^2 + Q_\rho^2 = P_\tau^2 + Q_\tau^2$$

1. We shall here introduce the symbol

$$(1) \qquad \left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\}$$

and define it as follows. An equality

$$(2) \qquad \left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\} = \left\{\frac{P_\tau + \sqrt{D}}{Q_\tau}\right\}$$

means that these two fractions, which do not need to be reduced, will on the formation of a chain lead to the same closed chain. Further, by

$$(3) \qquad \left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\} \left\{\frac{P_\tau + \sqrt{D}}{Q_\tau}\right\} = \left\{\frac{P_{\rho,\tau} + \sqrt{D}}{Q_\rho Q_\tau}\right\},$$

we mean that the two fractions to the left will be compounded into one single fraction according to Gauss's rule for composition of quadratic forms. Then we may consider the "ideal fraction" $\left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\}$ as a symbolic expression for the formation of the chain

$$P_\rho^2 - D = Q_{\rho-1} Q_\rho, \qquad P_\rho + P_{\rho+1} \equiv 0 (Q_\rho),$$
$$P_{\rho+1}^2 - D = Q_\rho Q_{\rho+1},$$
$$\text{etc.}$$

where, if desired, $P_\chi, Q_{\chi-1}, Q_\chi$ may represent also negative numbers. In consequence of this we express the inverse chain

$$P_\rho^2 - D = Q_\rho Q_{\rho-1}, \quad P_{\rho-1}^2 - D = Q_{\rho-1} Q_{\rho-2}, \quad \text{etc.}$$

by

$$\left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\}.$$

But we have in the previous paper* proved the inverse chain to be the complementary chain of the element $(P_\rho + \sqrt{D})/Q_\rho$, which may then be represented by the symbol $\left\{\frac{P_{\rho+1} + \sqrt{D}}{Q_\rho}\right\}$, or by $P_\rho + P_{\rho-1} \equiv 0 (Q_\rho)$, also by $\left\{\frac{-P_\rho + \sqrt{D}}{Q_\rho}\right\}$. Hence we have the relation

$$\left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\}^{-1} = \left\{\frac{-P_\rho + \sqrt{D}}{Q_\rho}\right\}, \tag{4}$$

verified by the fact that

$$\left\{\frac{-P_\rho + \sqrt{D}}{Q_\rho}\right\}\left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\} = \left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\}^{-1}\left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\}$$

must always give the principal chain. Now, to begin with, we are going to prove

THEOREM 1. *From the equality*

$$\left\{\frac{P_\rho + \sqrt{D}}{Q_\rho}\right\} = \left\{\frac{P_\tau + \sqrt{D}}{Q_\tau}\right\}$$

there follows the equation $M^2 - DN^2 = Q_\rho Q_\tau$, *where the solution must belong to the congruence-root* $P_{\rho,\tau}$ *which is constructed from* P_ρ *and* P_τ *by Gauss's rule for composition.*

We shall first prove the following lemmas.

LEMMA 1. *If by formation of the chain*

$$P_\rho^2 - D = Q_{\rho-1} Q_\rho, \qquad P_\rho + P_{\rho+1} = \mu_\rho Q_\rho,$$
$$P_{\rho+1}^2 - D = Q_\rho Q_{\rho+1},$$

* Arwin. loc. cit.

it follows that $Q_{\rho-1}$ and Q_ρ are both even, then the same statement must hold for all Q_τ of the chain.

For we have

$$Q_\rho Q_{\rho+1} = P_{\rho+1}^2 - D = (\mu_\rho Q_\rho - P_\rho)^2 - P_\rho^2 + Q_\rho Q_{\rho-1}$$

$$= Q_\rho(\mu_\rho^2 Q_\rho - 2\mu_\rho P_\rho + Q_{\rho-1})$$

$$Q_{\rho+1} = Q_\rho \mu_\rho^2 - 2\mu_\rho P_\rho + Q_{\rho-1}$$

whence the truth of this lemma. Thus if $D \equiv 2, 3\ (4)$, Q_{i-1} and Q_i cannot be even simultaneously, but with $D \equiv 1\ (4)$ this is possible, and we have, in fact, chains of fractions of two different types, which cannot be transformed into each other by any equivalence. If $D \equiv 5\ (8)$ Q_{i-1} and Q_i, being even, can only be divisible by the first power of 2.

LEMMA 2. *To each fraction $(P_\rho + \sqrt{D})/Q_\rho$ it is always possible to find an equivalent one $(P_\chi + \sqrt{D})/Q_\chi$, where Q_χ (or $Q_\chi/2$ in the case of the second type) is relatively prime to any arbitrarily given number, for example, Q_ρ itself (or $Q_\rho/2$).*

From

$$P_\rho^2 - D = Q_{\rho-1} Q_\rho, \quad P_{\rho+1}^2 - D = Q_\rho Q_{\rho+1}$$

let us form

$$(P_{\rho+1} + \mu Q_\rho)^2 - D = Q_\rho Q'_{\rho+1} = Q_\rho(\mu^2 Q_\rho + 2\mu P_\rho + Q_{\rho+1}).$$

Take $\mu = \mu_\nu$ determined by the incongruences $\mu_\nu \not\equiv 0\ (p_i)$ for all p_i common to Q_ρ and $Q_{\rho+1}$ ($Q_\rho/2$ and $Q_{\rho+1}/2$) and $\mu_\nu \not\equiv \mu_\tau\ (p_\tau)$ for all other p_τ in Q_ρ, where μ_τ are solutions of the congruence $2\mu_\tau P + Q_{\rho+1} \equiv 0\ (p_\tau)$. The infinity of the μ_ν give numbers $Q'_{\rho+1}$ (or $Q'_{\rho+1}/2$), which are required by Lemma 2.

We now prove Theorem 1. Form the chain

$$P_\rho^2 - D = Q_{\rho-1} Q_\rho, \quad P_{\rho+1}^2 - D = Q_\rho Q_{\rho+1}, \quad \ldots, \quad P_\tau^2 - D = Q_{\tau-1} Q_\tau,$$

and the inverse chain

$$(Q_\tau - P_\tau)^2 - D = Q_{\tau+1} Q_\tau, \; P_\tau^2 - D = Q_\tau Q_{\tau-1}, \ldots, \; P_{\rho+1}^2 - D = Q_{\rho+1} Q_\rho.$$

By means of a general formula* the former chain yields $Q_\rho Q_\tau = M_1^2 - D N_1^2$, where $N_1 = z_\chi$, $\pm M_1 = Q_\rho y_\chi - P_\rho z_\chi$ and the inverse chain $Q_\rho Q_\tau = M_2^2 - D N_2^2$, where $N_2 = z_\chi$, $\pm M_2 = Q_\tau (z_\chi - z_{\chi-1}) - (Q_\tau - P_\tau) z_\chi = P_\tau z_\chi - Q_\tau z_{\chi-1}$.† Thus by $z_\chi = N_1 = N_2$ we must have $M_1 = \pm M_2$. Taking the same sign we are lead to the following formulas

$$2 M_1 = Q_\rho y_\chi - Q_\tau z_{\chi-1} + (P_\tau - P_\rho) z_\chi, \tag{5'}$$

$$Q_\rho y_\chi + Q_\tau z_{\chi-1} = (P_\tau + P_\rho) z_\chi. \tag{5''}$$

It may readily be verified that this combination of signs must be true, first directly by formulas known from the previous paper,‡ and then also by simple specialization $\tau = \varrho + 1$, $y_\chi = \beta_\rho$, $z_\chi = 1$, $z_{\chi-1} = 0$, giving the known relation $\beta_\rho Q_\rho = P_\rho + P_{\rho+1}$. By means of the first expression in (5) it follows at once, d being the greatest common factor of Q_ρ, Q_τ, $P_\tau - P_\rho$,§ that $M_1 \equiv 0 (d/2)$ and thus $N_1 \equiv 0 (d/2)$, D having no square factors. In the case $D \equiv 2, 3 (4)$ Q_ρ and Q_τ cannot both be even unless M_1 and N_1 are even, whence a final common factor 2 may be divided into $Q_\rho Q_\tau$ according to Gauss's rule for composition. In the case of $D \equiv 1 (4)$ and fractions of the second type we have to operate with the values $Q_\rho/2$, $Q_\tau/2$, $(P_\tau - P_\rho)/2$,‖ whence their greatest common factor $d \not\equiv 2$ also gives $M_1 \equiv N_1 \equiv 0 (d)$. In the fractions of the first type Q_{i-1} and Q_i cannot both be even, as was proved in Lemma 1. By means of¶

$$(P_\rho^2 - D) z_\chi - Q_\rho (P_\rho - P_\tau) y_\chi = Q_\rho Q_\tau y_{\chi-1} \tag{6'}$$

$$Q_{\rho-1} z_\chi - (P_\rho - P_\tau) y_\chi = Q_\tau y_{\chi-1} \tag{6''}$$

we have once more (since $Q_\rho \equiv Q_\tau \equiv 0 (2)$, and hence, referring to Lemma 1, $Q_{\rho-1} \not\equiv 0 (2)$) $N_1 \equiv z_\chi \equiv 0 (2^\mu)$, whence $M_1 \equiv 0 (2^\mu)$, when Q_ρ, Q_τ and $(P_\rho - P_\tau)$ have the greatest common factor 2^μ. But inversely we cannot have $M_1 \equiv N_1 \equiv 0 (2^{\mu+1})$, since in this case (5) afford the congruences $Q_\rho y_\chi - Q_\tau z_{\chi-1} \equiv 0 (2^{\mu+2})$, $Q_\rho y_\chi + Q_\tau z_{\chi-1} \equiv 0 (2^{\mu+2})$, that is $Q_\rho \equiv 0 (2^{\mu+1})$ and $Q_\tau \equiv 0 (2^{\mu+1})$, since $y_\chi \equiv z_{\chi-1} \not\equiv 0 (2)$, whence from (6'') $P_\rho - P_\tau \equiv 0$

* Arwin, loc. cit., § 1, (10).

† Perron, O., Die Lehre von den Kettenbrüchen, S. 32.

‡ Arwin, loc. cit., § 1, (8'').

§ Arwin, loc. cit., § 1, (9).

‖ Sommer, J., Vorlesungen über Zahlentheorie, S. 220.

¶ Loc. cit., § 1, (9).

$(2^{\mu+1})$ contrary to our assumption. In the chains of the second type we must with the greatest common factor 2^μ of Q_ρ, Q_τ, $(P_\rho - P_\tau)$ obtain $M_1 \equiv N_1 \equiv 0\ (2^{\mu-1})$ $(\mu > 1)$. Having $Q_\rho/2^\mu \equiv Q_\tau/2^\mu \equiv 0\ (2)$ or $Q_\rho/2^\mu \equiv Q_\tau 2^\mu \not\equiv 0\ (2)$, we might by (5′) have $M_1 \equiv N_1 \equiv 0\ (2^\mu)$, but in the former case we should from (6″) obtain $(P_\rho - P_\tau)/2^\mu \equiv 0\ (2)$ contrary to our assumption. In the latter case it is in fact possible to have $M_1 \equiv N_1 \equiv 0\ (2^\mu)$, and it is necessarily so if $D \equiv 1\ (8)$ $\mu > 1$ or $Q_\rho/2 \equiv Q_\tau 2 \not\equiv 0\ (2)$, but, if $D \equiv 5\ (8)$, we obtain cases of $M_1 \equiv N_1 \equiv 0\ (2)$ as well as $M_1 \equiv N_1 \not\equiv 0\ (2)$, since we must now always have $Q_\rho/2 \equiv Q_\tau/2 \not\equiv 0\ (2)$. As to (5) a final common odd factor of Q_ρ/d, Q_τ/d and z_χ might also divide M_1, but from (6″) we conclude that each common factor of z_χ and Q_τ must also divide $P_\rho - P_\tau$, since y_χ and z_χ are relatively prime. As the definitive result we have $M_1 = d\,M_2$, $N_1 = d\,N_2$, $M_2^2 - D\,N_2^2 = Q_\rho\,Q_\tau/d^2$, where now N_2 and $Q_\rho\,Q_\tau/d^2$ are relatively prime with the exception of the chains of the second type in the above named case. From $M_2 \equiv P_\tau\,N_2\,(Q_\tau/d)$, $M_2 \equiv -P_\rho\,N_2\,(Q_\rho/d)$, $M_2 \equiv -P_{\rho,\tau}\,N_2\,(Q_\rho\,Q_\tau/d^2)$ it follows at once that $P_{\rho,\tau} \equiv -P_\tau\,(Q_\tau/d)$, $P_{\rho,\tau} \equiv P_\rho\,(Q_\rho/d)$ and also $(P_\rho - P_\tau)\,d \cdot P_{\rho,\tau} \equiv (D - P_\rho\,P_\tau)\,d\ (Q_\rho\,Q_\tau/d^2)$, deduced by $M_2 + P_\rho\,N_2 = Q_\rho/d \cdot y_\chi$, $M_2 - P_\tau\,N_2 = -Q_\tau/d \cdot z_{\chi-1}$, $M_2^2 - P_\rho\,P_\tau\,N_2^2 - M_2\,N_2\,(P_\tau - P_\rho) = Q_\rho\,Q_\tau/d^2 \cdot y_\chi\,z_{\chi-1}$, $M_2^2 = D\,N_2^2 + Q_\rho\,Q_\tau/d^2$, $N_2^2(D - P_\rho\,P_\tau) - M_2\,N_2(P_\tau - P_\rho) = -Q_\rho\,Q_\tau/d^2\,(y_\chi\,z_{\chi-1} + 1)$. Hence from $y_\chi\,z_{\chi-1} - y_{\chi-1}\,z_\chi = -1$, $z_\chi \equiv 0(d)$ we have $N_2^2\,(D - P_\rho\,P_\tau)/d - M_2\,N_2\,(P_\tau - P_\rho)/d = -Q_\rho\,Q_\tau/d\,(y_\chi\,z_{\chi-1} + 1)/d$ and finally from $M_2 \equiv -P_{\rho,\tau}\,N_2\,(Q_\rho\,Q_\tau/d^2)$ we determine $N_2^2\,(D - P_\rho\,P_\tau)\,d - N_2^2\,P_{\rho,\tau}\,(P_\rho - P_\tau)/d \equiv 0$, that is $P_{\rho,\tau}\,(P_\rho - P_\tau)/d \equiv (D - P_\rho\,P_\tau)/d\ (Q_\rho\,Q_\tau/d^2)$. Hence Theorem 1 is proved. On the other hand consider the relation $+Q_\rho\,Q_\tau = M_1^2 - D\,N_1^2$, where N_1 is relatively prime to $Q_\rho\,Q_\tau$ and $M_1 \equiv -N_1\,P_{\rho,\tau}\,(Q_\rho\,Q_\tau)$. Further, put $z_\chi = N_1$ and form by $P_\rho \equiv P_{\rho,\tau}(Q_\rho)\,y_\chi$ as an integer in $Q_\rho\,y_\chi - P_\rho\,z_\chi = M_1$. Then y_χ and z_χ are relatively prime and the continued fraction $y_\chi/z_\chi = \beta_1 - \cfrac{1}{\beta_2} - \ldots - \cfrac{1}{\beta_{\chi-1}}$ may be formed. Now let us take $(P_\rho + \sqrt{D})/Q_\rho$ as the initial element and form the chain with the β_i as coefficients, we then reach the last element $(P_\chi + \sqrt{D})/Q_\chi$ and hence* $(Q_\rho\,y_\chi - P_\rho\,z_\chi)^2 - D\,z_\chi^2 = Q_\rho\,Q_\chi$, whence $Q_\tau = Q_\chi$. Proceeding from the inverse chain we get $P_\chi \equiv P_\tau \equiv -P_{\rho,\tau}\,(Q_\tau)$, and then we have proved that, conversely, from $M_1^2 - D\,N_1^2 = Q_\rho\,Q_\tau$ it follows that

$$\{(-P_{\rho,\tau} + \sqrt{D})/Q_\tau\} = \{(P_\tau + \sqrt{D})/Q_\tau\}$$
$$= \{(P_{\rho,\tau} + \sqrt{D})/Q_\rho\} = \{(P_\rho + \sqrt{D})/Q_\rho\}.$$

* Loc. cit., § 1, (10).

We may prove this same theorem in another way as follows. From $(M_1-\sqrt{D}\,N_1)/Q_\rho\cdot(M_1+\sqrt{D}\,N_1)/Q_\tau=1$ and P_ρ, P_τ being solutions of the congruences $M_1\equiv -N_1P_\rho\ (Q_\rho)$, $M_1\equiv N_1P_\tau\ (Q_\tau)$ we have $M_1=-N_1P_\rho+mQ_\rho$, $M_1=N_1P_\tau-nQ_\tau$, $M_1^2=-mnQ_\rho Q_\tau$, $-mn\equiv 1\,(N_1)$, since $Q_\rho Q_\tau$ and N_1 are relatively prime. Hence

$$[-N_1\cdot(P_\rho+\sqrt{D})/Q_\rho+m]\ [-n+N_1\cdot(P_\tau+\sqrt{D})/Q_\tau]=1,$$

that is

$$\frac{P_\rho+\sqrt{D}}{Q_\rho}=\frac{m\dfrac{P_\tau+\sqrt{D}}{Q_\tau}-\dfrac{mn+1}{N_1}}{N_1\dfrac{P_\tau+\sqrt{D}}{Q_\tau}-n},\tag{7}$$

whence

$$\{(P_\rho+\sqrt{D})/Q\}=\{(P_\tau+\sqrt{D})/Q_\tau\}.$$

We shall now prove this theorem generally, when Q_ρ, Q_τ, $P_\rho-P_\tau$ have the greatest common factor d, but restated as follows.

THEOREM 2. *In any equivalence*

$$\{(P_\rho+\sqrt{D})/Q_\rho\}=\{(P_\tau+\sqrt{D})/Q_\tau\}\tag{8'}$$

we may on both sides compound with the arbitrarily given factor $\{(P_\nu+\sqrt{D})/Q_\nu\}$ *getting the equivalence*

$$\{(P_\nu+\sqrt{D})/Q_\nu\}\ \{(P_\rho+\sqrt{D})/Q_\rho\}=\{(P_\nu+\sqrt{D})/Q_\nu\}\ \{(P_\tau+\sqrt{D})/Q_\tau\}.\tag{8''}$$

We shall give the proof for the chains of the first type, since in chains of the second type we obtain the corresponding result by certain obvious changes, of which the most important is the substitution of Q_ρ and Q_τ for $Q_\rho/2$ and $Q_\tau/2$. Now, let us assume $Q_\nu=p_\nu^\lambda$ to be relatively prime to Q_τ, where p_ν is prime.

LEMMA 3. *From* (8') *we obtain for* $Q_\tau=Q_\rho$, $P_\tau=P'_\rho$ *the equivalence*

$$\begin{aligned}\{(P_\nu+\sqrt{D})/p_\nu^\lambda\}\ \{(P'_\rho+\sqrt{D})/p_\nu^\chi Q'_\rho\}&=\{(P'_\chi+\sqrt{D})/p_\nu^{\chi+\lambda-2\tau}Q'_\rho\}=\{(P_\chi+\sqrt{D})/p_\nu^{\lambda+\chi-2\tau}Q'_\rho\}\\&=\{(P_\nu+\sqrt{D})/p_\nu^\lambda\}\ \{(P_\rho+\sqrt{D})/p_\nu^\chi Q'_\rho\},\end{aligned}$$

with $P_\nu+P'_\rho=P_\nu+P_\rho\equiv 0\,(p_\nu^\tau)$ *for* $P'_\rho=a\,Q_\rho+P_\rho$ *and* $\chi>\tau$, $\lambda\geq\tau$.

According to Gauss's rule of compounding quadratic forms* we have $P'_\chi(P'_\rho+P_\nu)/p_\nu^\tau \equiv (D+P'_\rho P_\nu)/p_\nu^\tau (p_\nu^{\chi+\lambda-2\tau} Q'_\rho)$ or $P'_\chi(P_\rho+P_\nu)/p_\nu^\tau + a\, Q_\rho (P'_\chi - P_\nu)/p_\nu^\tau \equiv (D+P_\rho P_\nu)/p_\nu^\tau (p_\nu^{\chi+\lambda-2\tau} Q'_\rho)$ and also $P_\chi(P_\rho+P_\nu)/p_\nu^\tau \equiv (D+P_\rho P_\nu)\, p_\nu^\tau$, i. e., $(P'_\chi - P_\chi)(P_\rho+P_\nu)/p_\nu^\tau + a\, Q_\rho (P'_\chi - P_\nu)/p_\nu^\tau \equiv 0 (p_\nu^{\chi+\lambda-2\tau} Q'_\rho)$, hence $P'_\chi \equiv P_\chi (p_\nu^{\lambda+\chi-2\tau} Q'_\rho)$, since $P'_\chi \equiv P_\nu (p_\nu^{\lambda-\tau}) Q_\rho \equiv 0 (p_\nu^\chi)$.

Let us now consider the equivalence (8′) and determine P_ρ so that $P_\rho^2 - D \equiv 0 (p_\nu^\chi)$, $\not\equiv 0 (p_\nu^{\chi+1})$, P_ν odd. From $M^2 - DN^2 = Q_\rho Q_\tau$, $Q_\rho Q_\tau$ being relatively prime to N, $Q_\rho = p_\nu^\chi Q'_\rho$ and for example $\chi \geqq \lambda$, we have once more $M = Q_\rho y_\chi - P_\rho z_\chi = P_\tau z_\chi - Q_\tau z_{\chi-1}$, $z_\chi = N$. With $P_\nu^2 \equiv D (p_\nu^\lambda)$ we determine P'_ρ from the congruence $P'_\rho (P_\rho + P_\nu)/p_\nu^\tau \equiv (D + P_\rho P_\tau)/p_\nu^\tau (p_\nu^{\chi+\lambda-2\tau} Q'_\rho)$, when $P_\rho + P_\nu \not\equiv 0 (p_\nu^\tau)$ $\lambda > \tau$. Further, we obtain $P'_\rho (P_\rho + P_\nu)\, p_\nu^\tau \equiv (D + P_\rho P_\nu)\, p_\nu^\tau = P_\rho (P_\rho + P_\nu)\, p_\nu^\tau - Q'_\rho Q_{\rho\,-1} p_\nu^{\chi-\tau} (p_\nu^{\chi+\lambda-2\tau})$, that is $P'_\rho - P_\rho \equiv 0 (p_\nu^{\chi-\tau})$, $\not\equiv 0 (p_\nu^{\chi-\tau+1})$, since $\chi+\lambda-2\tau > \chi-\tau$. Let us in $M = Q_\rho y_\chi - P_\rho z_\chi$ substitute for P_ρ $P'_\rho = P_\rho + m p_\nu^{\chi-\tau} Q'_\rho$ $(m \not\equiv 0)$, we obtain $M = p_\nu^{\chi-\tau} Q'_\rho (p_\nu^\tau y_\chi + m z_\chi) - P'_\rho z_\chi$. For $\tau > 0$ we have $m z_\chi \not\equiv 0 (p_\nu)$, hence $y'_\chi = p_\nu^\tau y_\chi + m z_\chi$ and $z'_\chi = p_\nu^{\lambda-\tau} z_\chi$ are relatively prime, whence we can form the continued fraction $y'_\chi / z'_\chi = \beta_0 - \cfrac{1}{\beta_1 - \cdots - \cfrac{1}{\beta_{\chi-1}}}$. Proceeding from $(P'_\rho + \sqrt{D})/Q'_\rho p_\nu^{\chi+\lambda-2\tau}$ as the initial element we form the chain with these β_i and reach the last element $(P'_\tau + \sqrt{D})/Q'_\tau$. Hence we obtain $(Q'_\rho y'_\chi p_\nu^{\chi+\lambda-2\tau} - P'_\rho z'_\chi)^2 - D z'^2_\chi = Q'_\tau Q'_\rho p_\nu^{\chi+\lambda-2\tau}$ or $Q'_\tau Q'_\rho p_\nu^{\chi-\lambda} = Q_\tau Q'_\rho p_\nu^\chi$, $Q'_\tau = Q_\tau p_\nu^\lambda$ and from $M p_\nu^{\lambda-\tau} = P'_\tau z'_\chi - Q_\tau z_{\chi-1} p_\nu^\lambda$ we take $P'_\tau \equiv P_\tau (Q_\tau)$. Further, we have $P'^2_\tau \equiv P_\nu^2 (p_\nu^\lambda)$, that is, $P'_\tau \equiv P_\nu (p_\nu^\lambda)$, since $P'_\tau \equiv P'_\rho \equiv P_\nu (p_\nu^{\lambda-\tau})$. If $p_\nu^\lambda = 2^\lambda$, we have $M = 2^\chi Q'_\rho y_\chi - P_\rho z_\chi$, $M^2 = DN^2 + 2^\chi Q'_\rho Q_\tau \equiv P_\rho^2 N^2 (2^{\chi+1})$ and therefore $P_\rho^2 - D \equiv 0 (2^\chi)$, $\not\equiv 0 (2^{\chi+1})$, whence we conclude as before. If $\lambda \leqq \tau$ we have directly from $M^2 - DN^2 = Q'_\rho p_\nu^{\chi-\lambda} \cdot Q_\tau p_\nu^\lambda$ the equivalence $\{(P_{\rho,\tau} + \sqrt{D})/Q'_\rho p_\nu^{\chi-\lambda}\} = \{(-P_{\rho,\tau} + \sqrt{D})/Q_\tau p_\nu^\lambda\}$, where $-P_{\rho,\tau} \equiv P_\tau (Q_\tau)$, $P_{\rho,\tau} \equiv P_\rho (Q'_\rho p_\nu^\chi)$, whence $P_{\rho,\tau} + P_\nu \equiv 0 (p_\nu^\lambda)$, $-P_{\rho,\tau} \equiv P_\nu (p_\nu^\lambda)$. If $\tau = 0$ we have $M = Q'_\rho p_\nu^\chi (y_\chi + m z_\chi) - P'_\rho z_\chi$ with $y_\chi + m z_\chi \not\equiv 0 (p_\nu)$ also for $\chi = 0$, since $M = Q'_\rho p_\nu^{\chi+1} (y_\chi + m z_\chi)/p_\nu - P'_\rho z_\chi$ should give $M^2 - DN^2 =$

* Gauss, C. F., Untersuchungen über höhere Arithmetik (D. v. Maser, H.) 1899, S. 253.

$Q'_\rho p_\nu^{\chi-1}(Q_\tau/p_\nu)$, that is, $Q_\tau \equiv 0\,(p_\nu)$ contrary to our assumption. Thus $y'_\chi = y_\chi + mz_\chi$ and $z'_\chi = p_\nu^\lambda z_\chi$ are relatively prime, and we conclude as before. It is readily seen how proofs are modified in the case $\lambda > \chi$. Thus Theorem 2 is proved for all Q_ν which have no common factor with the greatest common factor of Q_ρ and Q_τ. But according to the above Lemma 2 we may always determine a Q_χ relatively prime to $Q_\rho\, Q_\tau$ and even to $Q_\nu Q_\rho Q_\tau$, affording the equivalence

$$\{(P_\rho + \sqrt{D})/Q_\rho\} = \{(P_\chi + \sqrt{D})/Q_\chi\} = \{(P_\tau + \sqrt{D})/Q_\tau\}.$$

This element inserted, we may apply the preceding demonstration, whence Theorem 2 is proved. For compounding chains of the second type the proof may be given as above in the case of p_ν odd and with a slight modification if $p_\nu = 2$, and so also when we compound chains of the second type with chains of the first type. We can also give this Theorem 2 the following interpretation: In any correspondence by equivalence common factors may be divided out and be suppressed. In this form, however, it is a corollary of Theorem 1.

There now follows at once

THEOREM 3. *Equivalent fractions compounded with equivalent fractions give equivalent fractions.*

For if

$$\{(P_\rho + \sqrt{D})/Q_\rho\} = \{(P_\tau + \sqrt{D})/Q_\tau\}$$

and

$$\{(P_\mu + \sqrt{D})/Q_\mu\} = \{(P_\nu + \sqrt{D})/Q_\nu\}$$

we have

$$(9)\quad \{(P_\rho + \sqrt{D})/Q_\rho\}\{(P_\mu + \sqrt{D})/Q_\mu\} = \{(P_\mu + \sqrt{D})/Q_\mu\}\{(P_\tau + \sqrt{D})/Q_\tau\}$$
$$= \{(P_\nu + \sqrt{D})/Q_\nu\}\{(P_\tau + \sqrt{D})/Q_\tau\}.$$

Hence the proof. We might also have borrowed this important theorem from Gauss's theory of quadratic forms;* since we have already established a one-to-one correspondence between fractions and quadratic forms.† Assuming $Q_\nu = 1$ in (9″) we have the corollary: Each chain compounded with the principal chain cannot lead into a chain other than the given one. Then it may be suitable to use the symbol

$$(10')\qquad \{1\}$$

* Sommer, J., Vorlesungen über Zahlentheorie, 1907, S. 218.

† Arwin, loc. cit.

for the principal chain $\left\{\begin{matrix} P_\chi + \sqrt{D} \\ 1 \end{matrix}\right\}$ of the first type and the symbol

$$\{1\}' \tag{10''}$$

for the principal chain of the second type. Further we have

THEOREM 4. *Each "ideal fraction" may in a unique manner be divided into "prime factors".*

For, if $Q_\rho = q_1^{\mu_1} \cdots q_i^{\mu_i}$, where q_τ are prime numbers, we get identically

$$\left\{\begin{matrix} P_\rho + \sqrt{D} \\ Q_\rho \end{matrix}\right\} = \left\{\begin{matrix} P_\rho + \sqrt{D} \\ q_1^{\mu_1} \end{matrix}\right\} \cdots \left\{\begin{matrix} P_\rho + \sqrt{D} \\ q_i^{\mu_i} \end{matrix}\right\} \tag{11'}$$

or, with $P_\rho = P_{\tau,\mu}\,(q_\tau^{\mu_\tau})$, $P_{\tau,\mu} \quad P_\tau\,(q_\tau)$ and from the symbol

$$\left\{\begin{matrix} P_{\tau,\mu} + \sqrt{D} \\ q_\tau^{\mu_\tau} \end{matrix}\right\} = \left\{\begin{matrix} P_\tau + \sqrt{D} \\ q_\tau \end{matrix}\right\}^{\mu_\tau},$$

also

$$\left\{\begin{matrix} P_\rho + \sqrt{D} \\ Q_\rho \end{matrix}\right\} = \left\{\begin{matrix} P_1 + \sqrt{D} \\ q_1 \end{matrix}\right\}^{\mu_1} \cdots \left\{\begin{matrix} P_i + \sqrt{D} \\ q_i \end{matrix}\right\}^{\mu_i}, \tag{11''}$$

verified by successive multiplication with $\left\{\begin{matrix} P_\rho + \sqrt{D} \\ q_\chi'' \end{matrix}\right\}$ $(\chi = 1, 2 \cdots \iota - 1, \iota + 1, \cdots i)$ yielding finally the identical expression for the chain on both sides. This relation (11'') may also be interpreted as the division of $\left\{\begin{matrix} P_\rho + \sqrt{D} \\ Q_\rho \end{matrix}\right\}$ into its simplest, its "prime factors". Having any fraction of the second type, we write $2Q_\rho$ and $2q_\tau^{\mu_\tau}$ instead of Q_ρ and $q_\tau^{\mu_\tau}$. With (11'') we have obtained a further analogy between the symbol $\left\{\begin{matrix} P_\rho + \sqrt{D} \\ 2^\varepsilon Q_\rho \end{matrix}\right\}$ and the symbol $(Q_\rho, (P_\rho + \sqrt{D})/2^\varepsilon)$ $(\varepsilon = 0 \text{ or } 1)$ of ideal numbers.

THEOREM 5. *Each chain must, by successive compounding with itself, necessarily give the principal chain.*

For, as seen, the number of closed chains is finite, and thus any chain compounded with itself, must give some certain chain more than once, that is

$$\left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\}^\mu=\left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\}^\nu$$

$\mu \mid \nu$, whence

$$\left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\}^{\mu-\nu}=\{1\},$$

and the theorem is proved.

THEOREM 6. *The lowest power to which a chain must be raised in order to lead into the principal chain is a factor of the number h of the closed chains.*

For by

$$\left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\}^i, \qquad (i=1,2\cdots r) \tag{10}$$

we have r different, not equivalent chains. If the number of closed chains is already exhausted, we have $r=h$, if not, we select a chain $\left\{\frac{P_\nu+\sqrt{D}}{Q_\nu}\right\}$, which is not in the set of (10). Further from

$$\left\{\frac{P_\nu+\sqrt{D}}{Q_\nu}\right\}\left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\}^i \qquad (i=1,2\cdots r)$$

we obtain a new set of r different chains, which are non-equivalent to each other and to the set of (10). Proceeding in this manner the number of closed chains must finally be exhausted and thus $h=m\cdot r$, whence Theorem 6 is proved.

It may be of some interest to show how the raising to powers can be effected in practice. Form, for example,

$$\frac{P_\rho+\sqrt{D}}{Q_\rho}=\mu_\rho-\frac{P_{\rho+1}-\sqrt{D}}{Q_\rho}=\mu_\rho-\frac{Q_\rho/2^\varepsilon}{\frac{P_{\rho+1}+\sqrt{D}}{Q_{\rho+1}}}, \tag{11}$$

where $\varepsilon=0$ for chains of the first type and $\varepsilon=1$ for the second type. $P_{\rho+1}$ will be determined from the congruence $-P_{\rho+1}(P_\rho+P_\rho)/2^\varepsilon \equiv (D+P_\rho^2)/2^\varepsilon\ (Q_\rho^2/2^\varepsilon)$ with the sign $-$ so as to have $P_\rho+P_{\rho+1}\equiv 0\ (Q_\rho)$.

Now we have $(Q_\rho \mu_\rho - P_\rho)^2 - D\,1^2 = (Q_\rho/2^\varepsilon)^2\, 2^\varepsilon\, Q_{\rho+1}$, leading to the equivalence

$$\left\{\frac{P_{\rho+1}+\sqrt{D}}{Q_{\rho+1}}\right\} = \left\{\frac{-P_{\rho+1}+\sqrt{D}}{Q_\rho^2/2^\varepsilon}\right\}$$

$$= \left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\}\left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\} = \left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\}^2.$$

After having reduced the fraction $(P_{\rho+1}+\sqrt{D})/Q_{\rho+1}$ and having found, for example, the principal chain, we conclude that the chain must belong to the exponent 2. If not, the operation in (11) can be repeated with some slight modification. We may, however, also proceed as follows. From $P_\rho^2 \equiv D\,(Q_\rho)$ we determine $P_{\rho+1}'^2 \equiv D\,(Q_\rho^3/2^{2\varepsilon})$ and have

$$\frac{P_\rho+\sqrt{D}}{Q_\rho} = \mu_\rho' - \frac{P_{\rho+1}'-\sqrt{D}}{Q_\rho} = \mu_\rho' - \frac{Q_\rho^2/2^{2\varepsilon}}{\dfrac{P_{\rho+1}'+\sqrt{D}}{Q_{\rho+1}'}},$$

whence

$$\left\{\frac{P_{\rho+1}'+\sqrt{D}}{Q_{\rho+1}'}\right\} = \left\{\frac{P_\rho+\sqrt{D}}{Q_\rho}\right\}^3.$$

I shall illustrate this with some simple examples.

Example. $D = 79$.

$$\frac{\sqrt{79}+8}{1} = 16 + \frac{\sqrt{79}-8}{1} = 16 + \frac{1}{\dfrac{\sqrt{79}+8}{15}},$$

$$\frac{\sqrt{79}+8}{15} = 1 + \frac{\sqrt{79}-7}{15} = 1 + \frac{1}{\dfrac{\sqrt{79}+7}{2}}.$$

Now form with $P_\rho = 7$, $Q_\rho = 3$ the congruence $-P_\rho \cdot 2 \cdot 7 \equiv 79 + 49$, $P_{\rho+1} \equiv 5\,(9)$. Hence

$$\frac{\sqrt{79}+7}{3} = 4 + \frac{\sqrt{79}-5}{3} = 4 + \frac{3}{\frac{\sqrt{79}+5}{6}},$$

$$\frac{\sqrt{79}+5}{6} = 2 + \frac{\sqrt{79}-7}{6} = 2 + \frac{1}{\frac{\sqrt{79}+7}{5}},$$

$$\frac{\sqrt{79}+7}{5} = 3 + \frac{\sqrt{79}-8}{5} = 3 + \frac{3}{\frac{\sqrt{79}+8}{1}},$$

where 8 will be determined by $-P_\tau(7+7) \equiv 79+49$, $P_\tau \equiv 8\ (15)$. Hence

$$\left\{\frac{\sqrt{79}+7}{3}\right\}^3 = \{1\} \tag{12'}$$

and further from $y_\chi/z_\chi = 4 + \frac{3|}{|2} + \frac{1|}{|3} = \frac{37}{7}$ we have $(3\cdot 37 - 7\cdot 7)^2 - 7^2\cdot 79 = -3^3$, which verifies (12'). We might, however, also have proceeded as follows. Starting from $P_{\rho+1}$ we determine $P'^2_{\rho+1} \equiv 79\ (3^3)$, leading in the above case to $P_{\rho+1} = P'_{\rho+1} = 5$, from which

$$\frac{\sqrt{79}+7}{3} = 4 + \frac{\sqrt{79}-5}{3} = 4 + \frac{3^2}{\frac{\sqrt{79}+5}{2}},$$

$$\frac{\sqrt{79}+5}{2} = 6 + \frac{\sqrt{79}-7}{2} = 6 + \frac{1}{\frac{\sqrt{79}+7}{15}},$$

$$\frac{\sqrt{79}+7}{15} = 1 + \frac{\sqrt{79}-8}{15} = 1 + \frac{1}{\frac{\sqrt{79}+8}{1}},$$

$$y_\chi/z_\chi = 4 + \frac{3^2|}{|6} + \frac{1|}{|1} = \frac{37}{7},$$

as already found.

Example. $D = 901$, $P_\rho = 29$, $Q_\rho = 6$, $-P_{\rho+1}\cdot 2\cdot 29 \equiv 901 + 29^2$. $P_{\rho+1} \equiv 1\ (18)$,

$$\frac{\sqrt{901}+29}{6} = 5 + \frac{\sqrt{901}-1}{6} = 5 + \frac{3}{\frac{\sqrt{901}+1}{50}},$$

$$\frac{\sqrt{901}+29}{6} = 8 + \frac{\sqrt{901}-19}{6} = 8 + \frac{3^2}{\frac{\sqrt{901}+19}{10}},$$

since $(1+18)^2 = 901$ (54). It may be verified that $(\sqrt{901}+19)/10$ will lead into the complementary chain of $(\sqrt{901}+29)/6$, hence

$$\left\{\frac{\sqrt{901}+29}{6}\right\}^3 = \left\{\frac{\sqrt{901}+19}{10}\right\} = \left\{\frac{\sqrt{901}+29}{6}\right\}^{-1}$$

that is

$$\left\{\frac{\sqrt{901}+29}{6}\right\}^4 = \{1\}', \tag{12''}$$

which again is verified by

$$\frac{\sqrt{901}+29}{6} = 26 - \frac{127-\sqrt{901}}{6} = 26 - \frac{3^3}{\frac{127+\sqrt{901}}{94}},$$

$$\frac{127+\sqrt{901}}{94} = 1 - \frac{-33-\sqrt{901}}{94} = 1 - \frac{1}{\frac{-33+\sqrt{901}}{2}}.$$

where $y_\chi/z_\chi = 26 - \frac{3^3}{1} = \frac{-1}{1}$, $(-6-29)^2 - 901\cdot 1^2 = 4\cdot 3^4$, whence follows the truth of (12″).

THEOREM 7. *If we assume the solution of Pell's equation and an arbitrary reduced fraction from any closed chain, the period of this closed chain may be computed without forming the chain itself.*

We shall prove this for negatively reduced chains. The relation $(Q_\rho y_\chi - P_\rho z_\chi)^2 - D z_\chi^2 = \pm Q^2$, where $(P_\rho + \sqrt{D})/Q_\rho$ is the initial as well as the last element of the closed chain, must, as we have seen,* give the solution of Pell's equation $M^2 - DN^2 = \pm 1$ in all cases of chains of the first type and also in the case of the second type, if $R^2 - DS^2 \neq \pm 4$. Hence $z_\chi = Q_\rho N$, $y_\chi = P_\rho N(\pm) M$, where the upper sign must be taken, since also in the case $1 > \sqrt{D}N - M > 0$ with $\sqrt{D} > P_\rho$ there follows

* Arwin, loc. cit.

$P_\rho N - M < 0$; and where y_χ, z_χ are relatively prime, since M and N are relatively prime, and since for $-M \equiv P_\rho N$, $Q_\rho \equiv 0\,(d)$ we should have $M^2 \equiv P_\rho^2 N^2 \equiv DN^2\,(d)$, $\pm 1 \equiv 0\,(d)$. Forming the continued fraction $y_\chi/z_\chi = \beta_0 + \cdots + \cfrac{1}{\beta_\chi}$ we have the period if we consider the sign of the right hand member of Pell's equation, which will determine this as odd or even.

Example. $D = 29$, $70^2 - 29 \cdot 13^2 = -1$, $(\sqrt{29}+2)/5$, leading to $z_\chi = 5 \cdot 13 = 65$, $y_\chi = 2 \cdot 13 + 70$. Now we have $96/65 = 1 + \cfrac{1}{2} + \cfrac{1}{10} + \cfrac{1}{8}$ and, since $N(\eta) = -1$, the period is odd; hence $96/65 = 1 + \cfrac{1}{2} + \cfrac{1}{10} + \cfrac{1}{2} + \cfrac{1}{1}$ with the period 5, yielding a symmetrical chain, as it should. We also remark that N from $M^2 - DN^2 = \pm 1$ must appear as a factor in every z_χ whether belonging to the same chain or not. It is readily seen how to modify this in the chains of the second type, when $M^2 - DN^2 = \pm 4$ and $D \equiv 5\,(8)$ with $Q_\rho \equiv 0\,(2)$, $Q_\rho \not\equiv 0\,(4)$.

Here it may be of some interest to call attention to the simple connection between the solution of $M^2 - DN^2 = \pm 4$, when existing, and those of $R^2 - DS^2 = \pm 1$. From $[(M + \sqrt{D}N)/2]^3 = R + \sqrt{D}S$ we have at once $M^3 + 3MN^2D = M^3 + 3M(M^2 \mp 4) = 8R$, $M^3 \mp 3M = 2R$, $3M^2N + DN^3 = N(3M^2 + M^2 \mp 4) = 8S$, $N(M^2 \mp 1) = 2S$. Hence

$$R/M - S/N = \mp 1. \tag{13}$$

Conversely we verify $[M(M^2 \mp 1) \mp 2M]^2 - DN^2(M^2 \mp 1)^2 = \pm 4$, $(M^2 \mp 1)^2(M^2 - DN^2 \mp 4) = 0$.

2. Let us have the chain of the second type $\{(P_\rho + \sqrt{D})/2Q_\rho\}$. If in $D - P_\rho^2 = 2Q_\rho \cdot 2Q_{\rho-1}$, where $D \equiv 1\,(8)$, Q_ρ is odd, we shall make correspond to this chain the chain $\{(P_\rho + \sqrt{D})/Q_\rho\}$ of the first type. It is readily seen that all chains $\{(P'_\rho + \sqrt{D})/4Q_\rho\}$, where $P'_\rho \equiv P_\rho\,(2Q_\rho)$, are of the second type. If Q_ρ is even, we can according to Lemma 1 determine P_ρ so that $Q_{\rho-1}$ will become odd, i. e., there exists necessarily a chain of the first type $\{(P'_\rho + \sqrt{D})/4Q_\rho\}$, where $P'_\rho \equiv P_\rho\,(2Q_\rho)$. If $\{(P_\rho + \sqrt{D})/4Q_\rho\}$ belongs to the first type, $Q_{\rho-1}$ must be odd; hence from $D - (P_\rho - 2Q_\rho)^2 = 2Q_\rho \cdot 2Q'_{\rho-1} = 2(Q_{\rho-1} + P_\rho - Q_\rho) \cdot 2Q_\rho$, we see that $Q'_{\rho-1}$ is even and therefore the chain $\{(P_\rho - 2Q_\rho + \sqrt{D})/4Q_\rho\}$ is of the second type, and so is also $\{(P_\rho + \sqrt{D})/Q_\rho\}$. If however $\{(P_\rho + \sqrt{D})/4Q_\rho\}$ belongs to the

second type, the chain $\{(P_\rho - 2Q_\rho + \sqrt{D})/4Q_\rho\}$ must be of the first type. For all Q_ρ we can therefore to the chain of the second type $\{(P_\rho + \sqrt{D})/2Q_\rho\}$ make correspond uniquely a certain chain of the first type, and conversely we return by compounding this chain of the first type with $\{(P_\rho + \sqrt{D})/2\}$ to the chain $\{(P_\rho + \sqrt{D})/2Q_\rho\}$. Hence we have established a one-to-one correspondence between the two types of chains in the case $D \equiv 1\ (8)$ after having proved that the above correspondence is independent of the representing element $(P_\rho + \sqrt{D})/2Q_\rho$. Let us therefore consider the equivalence

$$(14) \qquad \{\omega\} = \{(P_\rho + \sqrt{D})/2Q_\rho\} = \{(P_\tau + \sqrt{D})/2Q_\tau\} = \{\omega_1\},$$

that is, a relation $\omega = (y_\chi \omega_1 + y_{\chi-1})/(z_\chi \omega_1 + z_{\chi-1})$. If $Q_\rho = Q_\tau \not\equiv 0\ (2)$ we must have, as proved, $z_\chi \equiv 0\ (2)$, hence $2\omega = (y_\chi \cdot 2\omega_1 - 2y_{\chi-1})/(z_\chi/2\,2\omega_1 + z_{\chi-1})$, i. e., $2\omega \sim 2\omega_1$, as it should be. If $Q_\rho \not\equiv 0\ (2)$, $Q_\tau \equiv 0\ (2)$, we must have $z_\chi \not\equiv 0\ (2)$, since the chain $\{2\omega_1\}$ is now of the second type, as we have proved above; hence for $z_{\chi-1} \equiv 0\ (2)$ we find $2\omega = (2y_\chi \cdot \omega_1/2 + y_{\chi-1})/(z_\chi \cdot \omega_1/2 + z_{\chi-1}/2)$, that is $2\omega \sim \omega_1/2$, and for $z_{\chi-1} \not\equiv 0\ (2)$ $\omega_1 = \omega_3 + 1$ $2\omega = [2y_\chi \cdot \omega_3/2 + (y_{\chi-1} - y_\chi)]\,[z_\chi \cdot \omega_3/2 + (z_{\chi-1} - z_\chi)/2]$, that is, $2\omega \sim \omega_3/2$ conformably to our data. In the same way if $Q_\rho = Q_\tau \equiv 0\ (2)$, we can determine uniquely for all cases, how the elements are equivalent, constituting the chain of the first type. In the case $D \equiv 5\ (8)$, Q_ρ and Q_τ in (14) are necessarily odd, and all three chains $\{(P_\rho + \sqrt{D})/Q_\rho\}$, $\{(P_\rho + \sqrt{D})/4Q_\rho\}$, $\{(P_\rho - 2Q_\rho + \sqrt{D})/4Q_\rho\}$ are of the first type. If $M^2 - DN^2 = \mp 4$ is soluble in integers, they coincide, however, and give only the chain $\{(P_\rho + \sqrt{D})/Q_\rho\}$. Hence we have proved

THEOREM 8. *If $D \equiv 1\ (8)$ and $D \equiv 5\ (8)$ with $M^2 - DN^2 = \mp 4$ soluble, the number of closed chains of each type is equal.*

Now, having $M^2 - DN^2 = \mp 4$, the equivalences

$$(16) \qquad \{(P_\rho + \sqrt{D})/Q_\rho\} = \{(-P_\rho + \sqrt{D})/4Q_\rho\},$$

$$\{(-P_\rho + 2Q_\rho + \sqrt{D})/4Q_\rho\} = \{(P_\rho + \sqrt{D})/Q_\rho\}$$

give rise to relations $M^2 - DN^2 = \mp 4Q^2$, possible only when $\{(P_\rho + \sqrt{D})/2Q_\rho\}$ is symmetrical. From the equivalence $\{(P_\rho + \sqrt{D})/4Q_\rho\} = \{(-P_\rho + 2Q_\rho + \sqrt{D})/4Q_\rho\}$ we have $M^2 - DN^2 = \mp Q_\rho^2$. Now let us assume $\{(P_\rho + \sqrt{D})/2Q_\rho\}$ symmetrical with $D \equiv 0\ (Q_\rho)$, then $\{(P_\rho + \sqrt{D})/Q_\rho\}$ is

also symmetrical, but in this case the two chains $\{(P_\rho+\sqrt{D})/4Q_\rho\}$, $\{(-P_\rho+2Q_\rho+\sqrt{D})/4Q_\rho\}$ cannot be symmetrical. If $\{(P_\rho+\sqrt{D})/2Q_\rho\}$ is symmetrical so that with $N(\eta)=\mp 1$ we have $D=P_\rho^2+(2Q_\rho)^2$, we conclude from $D-(P_\rho-2Q_\rho)^2=P_\rho\cdot 4Q_\rho$ and the equivalence $\{(2Q_\rho+\sqrt{D})/P_\rho\}=\{(+P_\rho-2Q_\rho+\sqrt{D})/4Q_\rho\}$ this latter chain to be symmetrical; hence $\{(P_\rho+\sqrt{D})/Q_\rho\}$ cannot be symmetrical, since a relation $M^2-DN^2=\mp Q_\rho^2$, which must belong to the same congruence-root P_ρ as $D=P_\rho^2+(2Q_\rho)^2$ must suppose the solubility of $M^2-DN^2=\mp 4$. But also the chain $\{(P_\rho+\sqrt{D})/4Q_\rho\}$ cannot be symmetrical, since $\{(P_\rho-2Q_\rho+\sqrt{D})/4Q_\rho\}=\{(-P_\rho+2Q_\rho+\sqrt{D})/4Q_\rho\}$ must yield the relation $M^2-DN^2=\mp 4Q_\rho^2$ and therefore $\{(P_{\rho,\rho}+\sqrt{D})/Q_\rho\}=\{(-P_{\rho,\rho}+\sqrt{D})/4Q_\rho\}$. With $P_{\rho,\rho}=P_\rho(2Q_\rho)$ and $P_{\rho,\rho}=P_\rho-2Q_\rho(4Q_\rho)$ we should have $\{(P_\rho+\sqrt{D})/Q_\rho=\{(-P_\rho+2Q_\rho+\sqrt{D})/4Q_\rho\}$, which is not possible. Hence $P_{\rho,\rho}=P_\rho(4Q_\rho)$ and $\{(P_\rho+\sqrt{D})/Q_\rho\}=\{(-P_\rho+\sqrt{D})/4Q_\rho\}$, whence $\{(-P_\rho+\sqrt{D})/4Q_\rho\}$ cannot be symmetrical. This latter equivalence is also to be considered as the result of compounding the symmetrical chain $\{(P_\rho-2Q_\rho+\sqrt{D})/4Q_\rho\}=\{(-P_\rho+2Q_\rho+\sqrt{D})/4Q_\rho\}$ with $\{(-P_\rho+2Q_\rho+\sqrt{D})/4\}$. In the same way we have, after having compounded with $\{(P_\rho+\sqrt{D})/4\}$,* the equivalence $\{(P_\rho+\sqrt{D})/4Q_\rho\}=\{(-P_\rho+2Q_\rho+\sqrt{D})/4Q_\rho\}$. Hence we have shown that, when $\{(P_\rho+\sqrt{D})/2Q_\rho\}$ is non-symmetrical, none of the three corresponding chains is symmetrical, and also that all are distinct. If $\{(P_\rho+\sqrt{D})/2Q_\rho\}$ is symmetrical, one of the corresponding chains is symmetrical; the two other chains form complementary chains. It remains to show that the chain is independent of the choice of the particular representative element $(P_\rho+\sqrt{D})/2Q_\rho$. Now, in any equivalence $\omega=(y_\chi\omega_1+y_{\chi-1})/(z_\chi\omega_1+z_{\chi-1})$ we may have $z_\chi\equiv 0$ as well as $z_\chi\not\equiv 0\,(2)$. In the former case we obtain $2\omega\sim 2\omega_1$ and with, for example, $y_\chi\equiv 0\,(2)$ $\omega/2\sim\omega_1/2$, and for $\omega=\omega_2+1$, $\omega_1=\omega_3+1$ the equivalence $\omega_2/2\sim\omega_3/2$. If $y_{\chi-1}\not\equiv 0\,(2)$ we have $\omega/2\sim\omega_3/2$, $\omega_2/2\sim\omega_1/2$. And also in the case of $z_\chi\not\equiv 0\,(2)$ we may readily find the equivalences which determine in all cases the relations of the elements 2ω, $\omega/2$, $\omega_2/2$, $2\omega_1$, $\omega_1/2$, $\omega_3/2$. Hence we have proved the following theorem.

THEOREM 9. *If* $D\equiv 5\,(8)$ *and* $M^2-DN^2\neq\mp 4$ *the number of closed chains of the first type is thrice the number of closed chains of the second type.*

* That is, in the symmetrical chain $\{(P_\rho+\sqrt{D})Q_\rho\}=\{(-P_\rho+\sqrt{D})Q_\rho\}$.

Example. $D = 901$.

1. $$\frac{\sqrt{901}+29}{2} = 29 + \frac{\sqrt{901}-29}{2} = 29 + \frac{1}{\frac{\sqrt{901}+29}{30}},$$

$$\frac{\sqrt{901}+29}{30} = 1 + \frac{\sqrt{901}-1}{30} = 1 + \frac{1}{\frac{\sqrt{901}+1}{30}},$$

$$\frac{\sqrt{901}+1}{30} = 1 + \frac{\sqrt{901}-29}{30} = 1 + \frac{1}{\frac{\sqrt{901}+29}{2}},$$

2. $$\frac{\sqrt{901}+17}{34} = 1 + \frac{\sqrt{901}-17}{34} = 1 + \frac{1}{\frac{\sqrt{901}+17}{18}},$$

$$\frac{\sqrt{901}+17}{18} = 2 + \frac{\sqrt{901}-19}{18} = 2 + \frac{1}{\frac{\sqrt{901}+19}{30}},$$ Period 7.

etc.

3. $$\frac{\sqrt{901}+29}{6} = 9 + \frac{\sqrt{901}-25}{6} = 9 + \frac{1}{\frac{\sqrt{901}+25}{46}},$$

$$\frac{\sqrt{901}+25}{46} = 1 + \frac{\sqrt{901}-21}{46} = 1 + \frac{1}{\frac{\sqrt{901}+21}{10}},$$

$$\frac{\sqrt{901}+21}{10} = 5 + \frac{\sqrt{901}-29}{10} = 5 + \frac{1}{\frac{\sqrt{901}+29}{6}}.$$

With these chains we have to associate the following chains of the first type:

1^a. $$\frac{\sqrt{901}+29}{1} + 1 = \frac{\sqrt{901}+30}{1}$$

$$= 60 + \frac{\sqrt{901}-30}{1} = 60 + \frac{1}{\frac{\sqrt{901}+30}{1}},$$ Period 1.

$1^{\beta \text{ and } \gamma}$. $\dfrac{\sqrt{901}+29}{4} = 14 + \dfrac{\sqrt{901}-27}{4} = 14 + \dfrac{1}{\dfrac{\sqrt{901}+27}{43}}$, Period 3.

2^{α}. $\dfrac{\sqrt{901}+17}{17} = 2 + \dfrac{\sqrt{901}-17}{17} = 2 + \dfrac{1}{\dfrac{\sqrt{901}+17}{36}}$, Period 5.

$2^{\beta, \gamma}$. $\dfrac{\sqrt{901}+17}{68} = 1 + \dfrac{\sqrt{901}-51}{68} = 1 + \dfrac{1}{\dfrac{\sqrt{901}+51}{25}}$,

$\dfrac{\sqrt{901}+51}{25} = 3 + \dfrac{\sqrt{901}-24}{25} = 3 + \dfrac{1}{\dfrac{\sqrt{901}+24}{13}}$, Period 3.

etc.

3^{α}. $\dfrac{\sqrt{901}+29}{3} = 19 + \dfrac{\sqrt{901}-28}{3} = 19 + \dfrac{1}{\dfrac{\sqrt{901}+28}{39}}$, Period 3.

etc.

3^{β}. $\dfrac{\sqrt{901}+29}{12} = 4 + \dfrac{\sqrt{901}-19}{12} = 4 + \dfrac{1}{\dfrac{\sqrt{901}+19}{45}}$, Period 3.

etc.

3^{γ}. $\dfrac{\sqrt{901}+23}{12} = 4 + \dfrac{\sqrt{901}-25}{12} = 4 + \dfrac{1}{\dfrac{\sqrt{901}+25}{23}}$, Period 5.

Thus we have proved the theorem suggested in the previous paper: *If* $D \equiv 2, 3\ (4)$ *the number of closed chains equals that of the classes of ideals; if* $D \equiv 1\ (8)$, $D \equiv 5\ (8)$ *and* $M^2 - DN^2 = \pm 4$ *the number of closed chains will become double the number of the classes of ideals; and finally, if* $D \equiv 5\ (8)$ *with* $M^2 - DN^2 \neq \pm 4$ *this number will be four times the number of such classes, for in case* $D \equiv 1\ (4)$ *the number of ideal classes equals that of the closed chains of the second type.*

3. In this section we shall prove by means of our theory of closed chains the law of quadratic reciprocity, for which purpose we insert some simple lemmas.

Let p_ρ and q_ρ denote prime numbers $4n+1$ and $4n+3$, then the truth of this theorem will readily be verified: *The equation* $x^2 - p_\rho y^2 = \pm 2$ *is never, and* $x^2 - q_\rho y^2 = \pm 2$ *is always, soluble by integers.*

For if $D = p_\rho$, $Q_\nu = 2$ can never appear as denominator in the principal chain of the first type. If $D = q_\rho$, we have necessarily $N(\eta) = M^2 - DN^2 = +1$, as we already see from $M^2 - q_\rho N^2 \not\equiv -1\ (4)$, whence the middle element in the principal chain must be of the first category, hence $Q_\nu = 2$. We may also observe that $p_\rho = m^2 + n^2$, $N(\eta) = -1$, if $D = p_\rho$.

Further: If $D = \prod_{i=1}^{\tau} p_i$ and any equation of the form $p_\rho \cdots p_\chi x^2 - p_\mu \cdots \cdots p_\nu y^2 = \pm 1$ is soluble, then we must have $N(\eta) = +1$. For since $x^2 - Dy^2 = \pm p_\rho \cdots p_\chi < \pm \sqrt{D}\, p_\rho \cdots p_\chi$ must appear as denominator in the closed part of the principal chain, and hence as middle element, the period is even with $N(\eta) = +1$. Let us, for example, have $D = p_1 p_2$ with $(p_1/p_2) = -1$; then the relation $x^2 p_1 - y^2 p_2 = \pm 1$ cannot be soluble, whence $x^2 - p_1 p_1 y^2 = -1$. More important is the following theorem.

THEOREM 10. *If* $D = p_1, q_1, q_1 q_2$, *the number of closed chains of each type must be odd.*

For if $D = q_1$ we have necessarily $q_1 \not\equiv m^2 + n^2$, and, if $D = p_1$, we have just seen that $x^2 - p_1 y^2 \not\equiv \pm 2$, whence there can exist only one symmetric chain for $D = q_1$, and one of each type if $D = p_1$; and, as the non-symmetrical chains appear in pairs, the theorem is proved in these cases. Further, since $q_1 q_2 \equiv 1\ (4)$, the denominator $Q_\nu = 2$ cannot appear in the principal chains of the first type, and, since $q_1 q_2 \not\equiv m^2 + n^2$ the middle element as well as the initial must be of the first category, and hence there can only exist one symmetrical chain of the first type. It follows as a corollary that the equation $x^2 q_1 - y^2 q_2 = \pm 1$ is always soluble. Directly, but also by means of the above established one-to-one correspondence between the symmetric chains of the two types, we conclude that there exists only one symmetric chain of the second type. Thus Theorem 10 is completely proved.

THEOREM 11. *If* $(q_1/q_2) = +1$, *we have* $(q_2/q_1) = -1$ *and conversely.*

This follows immediately from the existence of the relation $x^2 q_1 - y^2 q_2 = \pm 1$.

THEOREM 12. *If* $(p_1/p_\rho) = +1$, *we have* $(p_\rho/p_1) = +1$ *and conversely.*

For with $D = p_1$ the number of closed chains must be odd, whence p_ρ, which appears as denominator in some chain, since $(p_1/p_\rho) = +1$, must leave, when raised to an odd power, a denominator in the principal chain; that is, the equation $x^2 - p_1 y^2 = \pm p_\rho (p_\rho)^{2\mu}$ must for some certain μ be soluble. Hence we have immediately $(p_\rho/p_1) = +1$. Reasoning in the same

manner for $D = p_\rho$ we may similarly prove the last part of the theorem. Thus theorem 11 is proved.

THEOREM 13. *If* $(p_1/q_\tau) = +1$, *we have* $(q_\tau/p_1) = +1$.

This theorem is proved in the same way as Theorem 12.

LEMMA 4: *If* $p_1 = 4 \cdot 2^r \cdot s + 1$, s *odd, it is always possible to find a* q_ν, *such that* $(p_1/q_\nu) = +1$.

For, assuming $(2n+1)^2 - p_1 = 4 \cdot 2^r \cdot M$ and $M \equiv 3\,(4)$ we may satisfy $n(n+1)\,2^r - s \equiv 3\,(4)$, and this is always possible since s is odd. From $M \equiv 3\,(4)$ we conclude the existence of at least one q_ν, which will give $(p_1/q_\nu) = +1$.

THEOREM 14. *If* $(q_\tau/p_1) = +1$, *we have* $(p_1/q_\tau) = +1$.

Let us take $D = q_\nu q_\tau$, q_ν determined as in Lemma 4. From the determined $(p_1 q_\nu) = +1$ we have according to Theorem 13 $(q_\nu/p_1) = +1$. As likewise $(q_\tau/p_1) = +1$, p_1 must appear as denominator of some fraction when $D = q_\nu q_\tau$, and, as the number of closed chains for this $D = q_\nu q_\tau$ is proved odd, the equation $x^2 - q_\nu q_\tau y^2 = (\pm) p_1 (p_1)^{2\chi}$ must be soluble for some certain χ. Comparing the condition $(p_1/q_\nu) = +1$ we see that the sign $+$ must be taken and thus conclude $(p_1/q_\tau) = +1$. Hence, with Theorems 11, 12, 13 and 14, the law of quadratic reciprocity is completely proved by means of the theory of closed chains.

4. In this section we shall treat the problem arising when $N(\eta) = M^2 - DN^2 = +1$ if $D \equiv 1, 2\,(4)$, D without any quadratic factors and containing only prime numbers $p_i = 4n+1$. We shall prove the following general theorem.

THEOREM 15. *The necessary and sufficient condition that* $N(\eta) = +1$ *when* $D \equiv 1\,(4)$ *is the existence of the equation*

$$(y_\chi^2 + y_{\chi-1}^2 + z_\chi^2 + z_{\chi-1}^2)^2 - (2^\varepsilon b)^2 D = 4, \qquad (\varepsilon = 0 \text{ or } 1) \tag{19}$$

which is satisfied by values $y_\chi, z_\chi, y_{\chi-1}, z_{\chi-1}$ *connected by the equation*

$$y_\chi z_{\chi-1} - y_{\chi-1} z_\chi = \pm 1, \tag{20}$$

and the congruence $y_\chi z_\chi + y_{\chi-1} z_{\chi-1} \equiv 0\,(b)$. y_χ/z_χ *also affords the equivalence*

$$\{(P_\rho + \sqrt{D})/Q_\rho\} = \{(P_\tau + \sqrt{D})/Q_\tau\} \tag{21}$$

where $D = P_\rho^2 + Q_\rho^2 = P_\tau^2 + Q_\tau^2$, $Q_\rho \equiv Q_\tau \equiv 0\,(2)$.

If $D \equiv 1\ (4)$, $N(\eta) = +1$, there must, as has been proved, exist symmetrical chains with elements of only the second category, hence $D = P_\rho^2 + Q_\rho^2 = P_\tau^2 + Q_\tau^2$. Let us fix $Q_\rho \equiv Q_\tau \equiv 0\ (2)$, whence $P_\rho \equiv P_\tau \not\equiv 0\ (2)$. Comparing equation (9) § 1 in the previous paper* we get for $D - P_\rho^2 = Q_\rho^2$, $z_{\chi-1} = -z_{\chi-1}$, $y_{\chi-1} = -y_{\chi-1}$ the following system

$$(22) \qquad Q_\tau y_{\chi-1} - Q_\rho z_\chi = (P_\rho - P_\tau) y_\chi, \quad Q_\rho y_\chi - Q_\tau z_{\chi-1} = (P_\rho + P_\tau) z_\chi.$$

Multiplying the left and the right hand members, respectively, of these equations we obtain

$$-Q_\tau^2 y_{\chi-1} z_{\chi-1} + Q_\rho Q_\tau (y_\chi y_{\chi-1} + z_\chi z_{\chi-1}) - Q_\rho^2 z_\chi y_\chi$$
$$= (P_\rho^2 - P_\tau^2) y_\chi z_\chi = (Q_\tau^2 - Q_\rho^2) y_\chi z_\chi.$$

Hence

$$(23) \qquad Q_\rho (y_\chi y_{\chi-1} + z_\chi z_{\chi-1}) = Q_\tau (y_\chi z_\chi + y_{\chi-1} z_{\chi-1}).$$

Now, a being the greatest common factor of Q_ρ and Q_τ, and b that of the parentheses, we conclude further that

$$(24') \qquad b Q_\rho = a (y_\chi z_\chi + y_{\chi-1} z_{\chi-1}), \quad b Q_\tau = a (y_\chi y_{\chi-1} + z_\chi z_{\chi-1})$$

and, from (22),

$$(24'') \quad b (P_\rho - P_\tau) = a (y_{\chi-1}^2 - z_\chi^2), \quad b (P_\rho + P_\tau) = a (y_\chi^2 - z_{\chi-1}^2),$$

or

$$(25) \quad 2b P_\rho = a (y_\chi^2 + y_{\chi-1}^2 - z_\chi^2 - z_{\chi-1}^2), \quad 2b P_\tau = a (y_\chi^2 + z_\chi^2 - y_{\chi-1}^2 - z_{\chi-1}^2).$$

Since P_ρ and Q_ρ are relatively prime, we have $2b \equiv 0\ (a)$ and may confine a to the values 1 or 2. From $D = P_\rho^2 + Q_\rho^2$ we compute

$$(26) \qquad (2^\varepsilon b)^2 D = 4 (y_\chi z_\chi + y_{\chi-1} z_{\chi-1})^2 + (y_\chi^2 + y_{\chi-1}^2 - z_\chi^2 - z_{\chi-1}^2)^2,$$

where $\varepsilon = 0$ if $a = 2$, $\varepsilon = 1$ if $a = 1$. According to $y_\chi z_{\chi-1} - y_{\chi-1} z_\chi = \mp 1$ there follows from (26)

$$(27) \qquad (2^\varepsilon b)^2 D = (y_\chi^2 + y_{\chi-1}^2 + z_\chi^2 + z_{\chi-1}^2)^2 - 4,$$

which is the relation of Theorem 15. But conversely from (27) the truth of (21) is readily shown. For from (27) we may arrive at (26), hence

* Loc. cit.

bQ_ρ and $2bP_\rho$ in (24′) and (25), since $y_\chi z_\chi + y_{\chi-1} z_{\chi-1} \equiv 0\,(b)$. Putting these values in the system (22) we compute bQ_τ and $2bP_\tau$ and have the expressions of (24′) and (25), also leading to (27). Now, since Q_τ and P_τ satisfy (22), it follows that

$$\{(P_\rho + D)/Q_\rho\} = \{(P_\tau + \sqrt{D})/Q_\rho\}, \tag{21}$$

hence $N(\eta) = +1$, and Theorem 15 is proved.

Next we shall investigate whether there exists any deeper relation between $P_\rho, Q_\rho, P_\tau, Q_\tau$, which are connected by the equivalence (21).

THEOREM 16. *In the equivalence* (21) *the values* $Q_\rho, Q_\tau, P_\rho, P_\tau$ *are connected by the following relations*

$$Q_\rho = am - bn, \quad Q_\tau = am + bn, \quad P_\rho = an + bm, \quad P_\tau = bm - an, \tag{28}$$

and are also determined by

$$\begin{aligned} Q_\rho + iP_\rho &= (a+ib)(m+in) = (am-bn) + i(an+bm), \\ Q_\tau + iP_\tau &= (a+ib)(m-in) = (am+bn) + i(bm-an), \end{aligned} \tag{29}$$

where $D = (a^2+b^2)(m^2+n^2)$.

Let us assume $p_\rho^\mu, p_\tau^{\nu_2}$ as powers of prime numbers in b if $a = 2$, or in $2b$ if $a = 1$, then, having $y_{\chi-1} \equiv z_\chi\,(p_\rho^\mu)$, $y_{\chi-1} \not\equiv 0\,(p_\rho)$, we obtain from (24′) $y_\chi z_\chi \equiv -y_{\chi-1} z_{\chi-1}$, $y_\chi \equiv z_\chi\,(p_\rho^\mu)$ and have the congruence $y_\chi + z_{\chi-1} \equiv 0\,(p_\rho^\mu)$, i. e., $y_\chi + z_{\chi-1} = p_\rho^\mu \cdot A$. Having $y_{\chi-1} \equiv z_\chi \equiv 0\,(p_\tau^{\nu_1})$ and, for example, by (24″) $y_\chi + z_{\chi-1} \equiv 0\,(p_\tau^{\nu_2})$, $y_\chi \equiv z_{\chi-1} \not\equiv 0\,(p_\tau)$, we find $(y_{\chi-1} - z_\chi)/p_\tau^{\nu_1} = p_\tau^{\nu_2-\nu_1} \cdot A$, if $\nu_2 > \nu_1$, and set $y_{\chi-1} - z_\chi = p_\tau^{\nu_2} \cdot A$ if $\nu_1 \geqq \nu_2$. Hence for all cases m, n, a, b are uniquely determined as integers by

$$y_\chi + z_{\chi-1} = r_1 m, \quad y_\chi - z_{\chi-1} = r_2 b, \quad y_{\chi-1} + z_\chi = r_2 a, \quad y_{\chi-1} - z = r_1 n$$

or

$$\begin{aligned} y_\chi &= (r_1 m + r_2 b)/2, \quad y_{\chi-1} = (r_1 n + r_2 a)/2, \\ z_\chi &= (r_2 a - r_1 n)/2, \quad z_{\chi-1} = (r_1 m - r_2 b)/2. \end{aligned} \tag{28}$$

Now, replacing in the system (24) $y_\chi, y_{\chi-1}, z_\chi, z_{\chi-1}$ by the values (28); we compute

$$\begin{aligned} bQ_\rho &= a/2 r_1 r_2 (ma - nb), \quad bQ_\tau = a/2 r_1 r_2 (ma + nb), \\ bP_\rho &= a/2 r_1 r_2 (na + mb), \quad bP_\tau = a/2 r_1 r_2 (mb - na), \end{aligned}$$

that is,

$$Q_\rho = ma - nb, \qquad Q_\tau = ma + nb,$$
$$P_\rho = na + mb, \qquad P_\tau = mb - na,$$
$$D = (a^2 + b^2)(m^2 + n^2) = D_1 D_2,$$

and likewise, replacing y_χ, $y_{\chi-1}$, z_χ, $z_{\chi-1}$ in (20) by the values (28), we have

$$r_1^2 (m^2 + n^2) - r_2^2 (a^2 + b^2) = r_1^2 D_1 - r_2^2 D_2 = \pm 4. \tag{29}$$

The existence of an equation like this may also be proved directly, for from

$$t^2 - (2^\varepsilon s)^2 D = 4 \qquad (\varepsilon = 0 \text{ or } 1) \tag{30}$$

we have, forming as in the previous paper

$$\left| \sqrt{t + 2^\varepsilon s \sqrt{D}} + \left| \sqrt{t - 2^\varepsilon s \sqrt{D}} \right. \right.$$
$$\left| \sqrt{t + 2^\varepsilon s \sqrt{D}} - \left| \sqrt{t - 2^\varepsilon s \sqrt{D}} \right. \right.,$$

since $M^2 - DN^2 = -4$ is not soluble, just an equation of type (29). But, conversely we can obtain (28) directly from (29) and hence from this equation, once given, construct all Q_ρ, Q_τ, P_ρ, P_τ satisfying (24). Thus Theorem 16 is proved.

Example. $D = 5 \cdot 53 \cdot 4241 = 1123865$

$$1^2 \cdot 4241 - 4^2 \cdot 265 = 1, \qquad b = 8, \qquad \varepsilon = 1.$$

Hence

$$1^2 (65^2 + 4^2) - 4^2 (12^2 + 11^2) = 1$$
$$(1 \cdot 65 + 4 \cdot 11)(1 \cdot 65 - 4 \cdot 11) - (4 \cdot 12 + 4 \cdot 1)(4 \cdot 12 - 4 \cdot 1) = +1,$$

whence

$$y_\chi = 109, \qquad z_\chi = 52, \qquad y_{\chi-1} = 44, \qquad z_{\chi-1} = 21.$$

Thus

$$8 Q_\rho = 109 \cdot 52 + 21 \cdot 44 = 8 \cdot 824,$$
$$8 Q_\tau = 109 \cdot 44 + 21 \cdot 52 = 8 \cdot 736,$$
$$16 P_\rho = 109^2 + 44^2 - 52^2 - 21^2 = 16 \cdot 667,$$
$$16 P_\tau = 109^2 + 52^2 - 44^2 - 21^2 = 16 \cdot 763,$$

and in fact

$$(2+i)\,(7+2i)\,(65+4i) = (12+11i)\,(65+4i) = 736+763i,$$

$$(2+i)\,(7+2i)\,(65-4i) = (12+11i)\,(65-4i) = 824+667i,$$

and also $y_\chi/z_\chi = 2+\dfrac{1}{|10}+\dfrac{1}{|2}+\dfrac{1}{|2}$, yielding the equivalence

$$\{(\sqrt{D}+667)/824\} = \{(\sqrt{D}+763)/736\},$$

where the intermediate fractions are $(\sqrt{D}+981)/196$, $(\sqrt{D}+979)/844$, $(\sqrt{D}+709)/736$.

From

$$(1\cdot 65+4\cdot 12)\,(1\cdot 65-4\cdot 12)-(4\cdot 11+4\cdot 1)\,(4\cdot 11-4\cdot 1) = 1,$$

we take $y_\chi = 113$, $z_\chi = 48$, $y_{\chi-1} = 40$, $z_{\chi-1} = 17$,

$$8Q_\rho = 8\,(113\cdot 6+17\cdot 5) = 8\cdot 763,$$

$$16P_\rho = 113^2+40^2-48^2-17^2 = 16\cdot 736,$$

so that the chain is of the first type.

If $D = 2\,(4)$, we have only one type of chain, whence the possibility

$$\{(P_\rho+\sqrt{D})\,Q_\rho\} = \{(Q_\rho+\sqrt{D})/P_\rho\} \tag{33}$$

is not to be excluded. In this case we have the following theorem:

THEOREM 17. *If $D = 2\,(4)$, $N(\eta) = +1$, then the equivalence-relation* (33) *exists when, and only when, $x^2-Dy^2 = \pm 2$ is soluble, otherwise we have* (21).

From $D-(Q_\rho-P_\rho)^2 = Q_\rho\cdot 2P_\rho$ we have

$$\{(P_\rho+\sqrt{D})/Q_\rho\} = \{(Q_\rho-P_\rho+\sqrt{D})/2P_\rho\}$$

$$= \{(Q_\rho-P_\rho+\sqrt{D})/2\}\,\{(Q_\rho+\sqrt{D})/P_\rho\} = \{(Q_\rho+\sqrt{D})/P_\rho\},$$

because $\{(Q_\rho-P_\rho+\sqrt{D})/2\} = \{1\}$, $x^2-Dy^2 = \pm 2$ is assumed to be soluble. Hence the theorem is proved. But it may also be demonstrated in the same way as Theorem 16. Placing $Q_\tau = P_\rho$ and $P_\tau = Q_\rho$ in (22) we find the following system of equations

$$Q_\rho(y_\chi-z_\chi) = P_\rho(y_\chi-y_{\chi-1}), \qquad Q_\rho(y_\chi-z_\chi) = P_\rho(z_\chi+z_{\chi-1}).$$

Since Q_ρ and P_ρ are relatively prime, we have

$$bQ_\rho = y_\chi - y_{\chi-1} = z_\chi + z_{\chi-1}, \qquad bP_\rho = y_\chi - z_\chi,$$

whence

$$b^2 D = P_\rho^2 + Q_\rho^2 = y_\chi^2 + y_{\chi-1}^2 - 2y_\chi y_{\chi-1} + y_\chi^2 + z_\chi^2 - 2y_\chi z_\chi =$$

$$= 2y_\chi (y_\chi - y_{\chi-1} - z_\chi) + y_{\chi-1}^2 + z_\chi^2 = 2y_\chi z_{\chi-1} + y_{\chi-1}^2 + z_\chi^2 =$$

$$= 2y_{\chi-1} z_\chi \pm 2 + y_{\chi-1}^2 + z_\chi^2 = (y_{\chi-1} + z_\chi)^2 \pm 2,$$

since $y_\chi z_{\chi-1} - y_{\chi-1} z_\chi = \pm 1$. Thus we have directly established the solubility of $x^2 - Dy^2 = \pm 2$ and also shown how to solve it.

Example. $D = 2 \cdot 17 \cdot 97 = 3298$.

Now we have $\{(\sqrt{D} + 56)/2\} = \{1\}$ from $y_\chi/z_\chi = 114 + \frac{1}{|2} + \frac{1}{|2} + \frac{1}{|1} = 801/7$. $(801 - 7 \cdot 57)^2 - D \cdot 7^2 = 2$, or $7^2(32^2 + 25^2) - 201^2(1^2 + 1^2) = +1$; hence

$$y_\chi = 425, \quad z_\chi = 376, \quad 7Q_\rho = 399, \quad Q_\rho = 57,$$

$$y_{\chi-1} = 26, \quad z_{\chi-1} = 23, \quad 7P_\rho = 49, \quad P_\rho = 7.$$

We have also $-7^2(40^2 + 7^2) + 201^2(1^2 + 1^2) = -1$,

$$y_\chi = 481, \quad z_\chi = 250, \quad 7Q_\rho = 329, \quad Q_\rho = 47,$$

$$y_{\chi-1} = 152, \quad z_{\chi-1} = 79, \quad 7P_\rho = 231, \quad P_\rho = 33,$$

$$3298 = 33^2 + 47^2,$$

and $\{(\sqrt{D} + 33)/47\} = \{(\sqrt{D} + 47)/33\}$ given by $481/250 = 1 + \frac{1}{|1} + \frac{1}{|12} + \frac{1}{|6} + \frac{1}{|3}$.

Example. $D = 2 \cdot 41 \cdot 73 = 5986 = 75^2 + 19^2 = 69^2 + 35^2$. Here the equivalence $\{(\sqrt{D} + 75)/19\} = \{(\sqrt{D} + 69)/35\}$ is satisfied by $18982/2367 = 8 + \frac{1}{|51} + \frac{1}{|2} + \frac{1}{|5} + \frac{1}{|4}$, $y_{\chi-1}/z_{\chi-1} = 4539/566$. The intermediate

fractions are $(\sqrt{D}+77)/3$, $(\sqrt{D}+76)/70$, $(\sqrt{D}+64)/27$, $(\sqrt{D}+71)/35$. We also compute

$$bQ_\rho = y_\chi z_\chi + y_{\chi-1} z_{\chi-1} = 2499972 \cdot 19,$$

$$2bP_\rho = y_\chi^2 + y_{\chi-1}^2 - z_\chi^2 - z_{\chi-1}^2 = 2499972 \cdot 75 \cdot 2,$$

and the equation $x^2 - 5986\,y^2 = \pm 2$ is not soluble.

With slight modification we can apply these theorems to positively closed chains. We have, for example, $D = 165$, $15 \cdot 1^2 - 11 \cdot 1^2 = 4$, $1^2(6^2 - 5^2) - 1^2(4^2 - 1^2) = -4$, $(6 \cdot 1 + 4 \cdot 1)(6 \cdot 1 - 4 \cdot 1) - (5 \cdot 1 + 1 \cdot 1)(5 \cdot 1 - 1 \cdot 1) = -4$, $y_\chi = 5$, $z_\chi = 3$, $y_{\chi-1} = 2$, $z_{\chi-1} = 1$, $y_\chi z_{\chi-1} - y_{\chi-1} z_\chi = -1$, $y_\chi : z_\chi = 5 : 3 = 2 - \cfrac{1}{3}$, and

$$bQ_\rho = 2(5 \cdot 3 - 2 \cdot 1) = 1 \cdot 26, \qquad bQ_\tau = 2(5 \cdot 2 - 3 \cdot 1) = 1 \cdot 14,$$

$$bP_\rho = 5^2 + 3^2 - 2^2 - 1^2 = 1 \cdot 29, \quad bP_\tau = 5^2 - 3^2 + 2^2 - 1^2 = 1 \cdot 19,$$

$$\{(\sqrt{165}+29)/26\} = \{(\sqrt{165}+19)/14\}.$$

Also in the case of an entire period, that is, $Q_\tau = Q_\rho$, $P_\tau = P_\rho$, this same method is applicable as may be seen from the following example.

Example. $D = 1105 = 5 \cdot 13 \cdot 17$, $D = 32^2 + 9^2 = 33^2 + 4^2 = 23^2 + 24^2 = 31^2 + 12^2$, $28488^2 - D \cdot 857^2 = -1$. Hence $(28488 + 9 \cdot 857)(28488 - 9 \cdot 857) - (32 \cdot 857)(32 \cdot 857) = -1$, $y_\chi / z_\chi = 36201/27424 = 1 + \cfrac{1}{3} + \cfrac{1}{8} + \cfrac{1}{33} + \cfrac{1}{8} + \cfrac{1}{3} + \cfrac{1}{1}$, $Q_\rho = 32$, $P_\rho = 9$, period 7, and from $(28488 + 32 \cdot 857)(28488 - 32 \cdot 857) - (9 \cdot 857)(9 \cdot 857) = -1$, $y_\chi : z_\chi = 55912/7713 = 7 + \cfrac{1}{4} + \cfrac{1}{66} + \cfrac{1}{4} + \cfrac{1}{7}$, $Q_\rho = 9$, $P_\rho = 32$, period 5. In the same way we may treat the other combinations of Q_ρ and P_ρ.

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ON THE CHARACTERIZATION OF THE SET OF POINTS OF λ-CONTINUITY.*

By Henry Blumberg.

The theorem of the present paper holds for one-valued, real functions of one or more variables; for the sake of greater concreteness, however, we confine the discussion to functions $z = g(x, y)$ of two variables. We use the term "I-region" (= open set) to denote a (planar) set consisting entirely of inner points, i. e., of points each of which is enclosable in a circle containing exclusively points of the set. It is known that the set of points where a function is continuous is a $\prod I_n$, i. e., a product (= common part) of a sequence of I-regions I_n; conversely, as W. H. Young has proved for the case of a single variable,† if S is a $\prod I_n$, a function exists that is continuous at every point of S and discontinuous elsewhere. Now to say that a function $g(x, y) = g(P)$ is continuous at $P = (x, y)$ is to say that the ordinary saltus of g is zero at P. This form of statement suggests other kinds of continuity. For besides the ordinary saltus, we have the f-saltus, the d-saltus, the z-saltus, and the e-saltus.‡

We are thus led to introduce the following

Definition. A function $g(P)$ is said to be λ-continuous at P, if its λ-saltus is zero at P; λ-discontinuous at P, if its λ-saltus is positive at P. Here λ is to be replaced by f, d, z, or e.

The object of the present paper is to prove the following

Theorem. *The points where a function is λ-continuous, $\lambda = f, d, z,$ or e, constitute a $\prod I_n$, i. e., a product of I-regions; conversely, if S is a $\prod I_n$, a function exists that is λ-continuous at every point of S and λ-discontinuous elsewhere.*

Let S be the set of points of λ-continuity. That S is a $\prod I_n$ follows at

* Read before the American Mathematical Society, December, 1919.

† "Über die Einteilung der unstetigen Funktionen und die Verteilung ihrer Stetigkeitspunkte," Wiener Sitzungsberichte, vol. CXII, Abt. IIa, p. 1307.

‡ These are defined as the ordinary saltus, except that in calculating them, we may neglect finite sets, denumerable sets, sets of zero measure (Lebesgue), and exhaustible sets (= sets of first category according to Baire) respectively. For the precise definition of the various saltus functions, see Blumberg, "Certain General Properties of Functions," these Annals, second series, vol. XVIII (1917), p. 147. These definitions are stated for functions of one variable, but they may be at once extended to the case of n variables.

once. For the λ-saltus is an upper semi-continuous, non-negative function.* Therefore, the points where the λ-saltus $< \frac{1}{n}$, $n = 1, 2, \ldots$, constitute an I-region I_n; and $S = \prod I_n$.

To prove the converse, we take first the case of z-continuity. We construct a function $h_1(x, y) = h_1(P)$ with the following properties: (a) $h_1(P) = 0$, if P is not in I_1; (b) $h_1(P)$ is continuous at every point of I_1; (c) $h_1(P) \leq 1/3$ everywhere; and (d) $h_1(P)$ assumes both the values $1/3$ and $-1/3$ in every neighborhood of every boundary point of I_1. Let

$$Q_1, Q_2, \ldots \text{ in inf.},$$

and

$$R_1, R_2, \ldots \text{ in inf.}$$

be two sequences of points of I_1, such that on the one hand, every point of the boundary of I_1 is a limit point both of the set $\{Q_\nu\}$ and of the set $\{R_\nu\}$, and on the other hand, every point of I_1 is a limit point of neither set. We define $h_1(Q_\nu)$ as equal to $1/3$ and $h_1(R_\nu)$ as equal to $-1/3$, for every ν. If P is a point outside of I_1, we set $h_1(P) = 0$. Finally, if P is a point of I_1, different from every Q_ν and R_ν, let $d(P)$ be the shortest distance from P to the closed set C that consists of all the points Q_ν and the points R_ν and all the boundary points of I_1; let V be a variable point that ranges over C, and PV, the distance from P to V. We then define† $h_1(P)$, V varying over C, by the equation

$$h_1(P) = min\left[h_1(V) + \frac{PV}{d(P)} - 1\right].$$

Let P_1 and P_2 be two P's of I_1 for which h_1 is defined by means of this equation; and at distance ϵ_1 apart; let ϵ_2 be an arbitrary positive number, and V, a point of C such that

$$h_1(P_1) > h_1(V) + \frac{P_1V}{d(P_1)} - 1 - \epsilon_2.$$

Since

$$P_2V \leq P_1V + \epsilon_1 \text{ and } d(P_2) \geq d(P_1) - \epsilon_1,$$

* L. c. p. 151. The reasoning is essentially the same for the case of a function of n variables.

† Cf. Hausdorff, "Über halbstetige Funktionen und deren Verallgemeinerung", Math. Zeitschrift, vol. 5 (1919), p. 296.

we have

$$h_1(P_2) \leqq h_1(V) + \frac{P_2V}{d(P_2)} - 1$$

$$\leqq h_1(V) + \frac{P_1V}{d(P_1)} - 1 + k\varepsilon_1,$$

where

$$k = \frac{d(P_1) + P_1V}{d(P_1)\,(d(P_1) - \varepsilon_1)}.$$

Therefore

$$h_1(P_2) - h_1(P_1) < k\varepsilon_1 + \varepsilon_2,$$

from which, in conjunction with the corresponding inequality for $h_1(P_1) - h_1(P_2)$, we may conclude that $h_1(P)$ is continuous at every point P of I_1 that does not belong to $\{Q_\nu\}$ or to $\{R_\nu\}$. Moreover, if P is sufficiently near a point Q_ν, V must be taken to coincide with Q_ν, if we are to obtain the minimum value of $h_1(V) + \frac{PV}{d(P)} - 1$. Hence $h_1(P) = h_1(Q_\nu)$ if P is sufficiently near Q_ν, and therefore h_1 is continuous at Q_ν. Likewise h_1 is continuous at every R_ν. Furthermore let V be a point of C at the distance $d(P)$ from the point P, which belongs to I_1, but not to $\{Q_n\}$ or $\{R_n\}$. Then

$$|h_1(P)| \leqq \left|h_1(V) + \frac{PV}{d(P)} - 1\right| = |h_1(V)| \leq 1/3.$$

The function $h_1(P)$ therefore possesses the properties (a), (b), (c), and (d).

We now define in the same way a function $h_n(P)$, $n = 1, 2, \ldots$, such that (a) $h_n(P) = 0$ if P is not in I_n; (b) $h_n(P)$ is continuous at every point of I_n; (c) $|h_n(P)| \leqq \frac{1}{3^n}$ everywhere; and (d) $h_n(P)$ assumes both the values $\frac{1}{3^n}$ and $-\frac{1}{3^n}$ in every neighborhood of every boundary point of I_n. Let

$$h = h_1 + h_2 + \cdots + h_n + \cdots.$$

The function h is continuous at every point P of $S = \coprod I_n$. For the function $h_1 + h_2 + \cdots + h_n$ is continuous at P, each of its terms being continuous at P; and the function $h_{n+1} + h_{n+2} + \cdots$, which is in absolute value $\leq \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} + \cdots$, approaches zero uniformly as $n \to \infty$.

If P is a point of $\bar{S}$, the complement of S, we shall say that "P is of index n" if it is "shed" at the nth stage, i. e., if I_n is the first I to which P does not belong. Suppose that P is of index n and belongs to the derivative S' of S. Since P belongs to every I_ν for $\nu < n$, the function $h_1 + h_2 + \cdots + h_{n-1}$ is continuous at P; and since P, being a limit point of S, is a boundary point of I_n, every neighborhood of P contains points of I_n at which $h_n = 1/3^n$ and also points of I_n at which $h_n = -\frac{1}{3^n}$. Moreover, since h_n is continuous at every point of I_n, the saltus of h_n is $2/3^n$ in every neighborhood of P even though sets of zero measure may be neglected. Therefore the z-saltus of h_n is $2/3^n$ at P. Since $h_1 + h_2 + \cdots + h_{n-1}$ is continuous at P, the z-saltus of $h_1 + h_2 + \cdots + h_n$ is $2/3^n$ at P. Now

$$|h_{n+1} + h_{n+2} + \cdots| \leq \frac{1}{3^{n+1}} + \frac{1}{3^{n+2}} + \cdots = \frac{1}{2 \cdot 3^n}.$$

Hence the z-saltus of h is at least $1/3^n$ at P. The function h is therefore z-discontinuous at P.

We finally define the function $g(P)$ as follows. We divide the set of interior points of $\bar{S}$ into 2 parts, T_1 and T_2, in such a way that every circle containing an interior point of $\bar{S}$ contains a subset of T_1 and also a subset of T_2, neither of which is of measure zero. We then define $g(P)$ as equal to $h(P)$ if P does not belong to T_1, and as equal to $h(P) + \varrho(P)$, if P belongs to T_1; here $\varrho(P)$ is the shortest distance from P to $S + S'$. We assert that $g(P)$ possesses the required property of being z-continuous—in fact, continuous—at every point of S, and z-discontinuous at every point of S. For if A is a point of $S + S'$, $|g(P) - h(P)|$ approaches 0 as P approaches A. Therefore $g(P)$ like $h(P)$ is continuous at every point of S, and z-discontinuous at every point of $\bar{S}S'$, the common part of $\bar{S}$ and S'. If A is an interior point of S, suppose first that $h(P)$ is continuous at A. It follows, according to the definition of $g(P)$, that every neighborhood of A has a z-saltus of at least $\varrho(A)$; $g(P)$ is therefore z-discontinuous at A. Finally, suppose that $h(P)$ is discontinuous at A. We may assume without loss of generality that $I_1 \supseteq I_2 \supset I_3 \cdots$, i. e., that I_1 contains I_2, I_2 contains I_3, and so on. For if this assumption is not fulfilled at the the start, we may set $J_n = \prod_{\nu=1}^{\nu=n} I_\nu$; then $S = \prod_{\nu=1}^{\nu=\infty} J_\nu$ with $J_1 \supset J_2 \supseteq \cdots$. Let the index of A be n. Then A must be a boundary point of I_n. For otherwise $h_n(P)$ would be zero for $\nu \geq n$ if P is in a sufficiently small neighborhood of A; since A is in I_{n-1}, the function $\sigma_{n-1} = h_1 + h_2 + \cdots + h_{n-1}$ is continuous at A,

and consequently h would be continuous at A contrary to our present assumption. Since A is a boundary point of I_n, every neighborhood of A has two points of I_n at which $h_n = 1/3^n$ and $-1/3^n$ respectively. Since σ_{n-1} is continuous at A the saltus of σ_{n-1} in a sufficiently small neighborhood N of A is less than a preassigned positive number ϵ. Let Q and R be two points in N at which $h_n = 1/3^n$ and $-1/3^n$ respectively. Since h_n is continuous at Q and at R, we may enclose Q and R in neighborhoods N_1 and N_2 respectively, both lying in N, such that the saltus of h_n is $< \epsilon$ in both N_1 and N_2. If P_1 and P_2 are any 2 points in N_1 and N_2 respectively, we have

$$\sigma_{n-1}(P_1) - \sigma_{n-1}(P_2) > -\epsilon \text{ and } h_n(P_1) - h_n(P_2) > \frac{2}{3^n} - 2\epsilon;$$

and since $|h_{n-1} + h_{n-2} + \cdots| < \frac{1}{2 \cdot 3^n}$, we conclude that $h(P_1) - h(P_2) > \frac{1}{3^n} - 3\epsilon$. The neighborhoods N_1 and N_2 contain respectively subsets M_1 and M_2 of $\overline{T}_1$—the complement of T_1—that are not of measure zero. Since g and h have the same values in M_1 and M_2 we conclude that the z-saltus of g is at least $1/3^n - 3\epsilon$ and hence at least $1/3^n$ in every sufficiently small neighborhood N of A. Therefore, the z-saltus of g is at least $1/3^n$ at A. The theorem is now completely proved for the z-saltus.

The construction of a suitable example in the case of the e-saltus may be carried out precisely in the same way, except that T_1 and T_2 are to be so chosen that every circle containing an interior point of S contains subsets both of T_1 and of T_2 that are inexhaustible (i. e., not of first category, according to Baire).

For the f- and the d-saltus, we may use the function g defined above, without modification. For since g is continuous at every point of S, it is *a fortiori* f-continuous and d-continuous at every point of S; and since g is z-discontinuous at every point of S, it is *a fortiori* f-discontinuous and d-discontinuous at every point of S.

UNIVERSITY OF ILLINOIS.

ON THE AMPERE-CAUCHY DERIVED FUNCTIONS.

By H. L. Smith.

According to Cauchy,* Ampere was the first to show how to derive from a given function $f(x)$ the sequence $f^{[1]}(x_0\, x_1)$, $f^{[2]}(x_0\, x_1\, x_2)$, ... of functions which we call in this paper the Ampere-Cauchy functions derived from $f(x)$. These functions were first defined for distinct values of the arguments involved. Then if the limit of one of them, say $f^{[n]}$, existed as certain of its arguments approached a common limiting value, that limit was taken as the defined value of $f^{[n]}$ when those arguments had assumed their common limiting value. The formula

$$f^{[n]}(x_0 \cdots x_n) = \frac{f^{[n]}(\xi)}{n!},$$

where ξ is between the smallest and the greatest of the distinct values $x_0, \ldots, x_n$, made it possible to identify $f^{[n]}\left(\overset{0}{a} \cdots \overset{n}{a}\right)$ with $f^{[n]}(a)/n!$ provided $f^{[n]}(x)$ was known to be continuous for $x = a$.

In the present paper a definition of the functions $f^{[n]}$ is given which allows us to compute their values without unnecessary restrictions as to continuity. The most that is required is the mere existence of all derivatives to which we are led. This leads to a new treatment of Newton's interpolation formula with remainder term under hypotheses which reduce, when the parameters involved are taken equal, to those under which Taylor's formula with a remainder term is proved in the calculus.†

1. **The Ampere-Cauchy derived functions (distinct arguments).** Any function $f(x)$ of a single variable x gives rise to an associated function $f^{[1]}(x_0\, x_1)$ of two variables defined for all distinct values of those variables by the equation

$$f^{[1]}(x_0\, x_1) = \frac{f(x_0) - f(x_1)}{x_0 - x_1}. \tag{1}$$

By induction we can define an infinite sequence $f^{[1]}, f^{[2]}, f^{[3]}, \ldots$ of functions derived from f. Thus on assuming that $f^{[1]}, f^{[2]}, \ldots, f^{[n-1]}$ have been

* Comptes Rendus, vol. 11.

† Other treatments equally general have already been given from other points of view. Cf. Markoff, Differenzenrechnung, Kap. 1; Birkhoff, Trans. Am. Math. Soc., vol. 7 (1909).

defined as functions of $2, 3, \ldots, n$ distinct arguments, respectively, we can then define $f^{[n]}$ by the equation

$$(2) \quad f^{[n]}(x_0, \ldots, x_n) = \frac{f^{[n-1]}(x_0, \ldots, x_{n-2}, x_{n-1}) - f^{[n-1]}(x_0, \ldots, x_{n-2}, x_n)}{x_{n-1} - x_n}$$

for all distinct values of $x_0, \ldots, x_n$. The functions $f^{[1]}$, $f^{[2]}$, $f^{[3]}$, ... are called the Ampere-Cauchy functions of orders 1, 2, 3, ..., respectively, derived from the function f.

By induction the formula

$$(3) \quad f^{[n]}(x_0, \ldots, x_n) = \sum_{i=0}^{n} \frac{f(x_i)}{(x_i - x_0) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}$$

is easily proved to hold for all distinct values of $x_0, \ldots, x_n$. Formula (3) shows that $f^{[n]}$ is a symmetric function of its $(n+1)$ arguments.

Before proceeding to extend the above definition to the general case where the arguments are not supposed distinct, we need a certain formula for difference quotients. This is the subject of the next section.

2. **A formula for difference quotients.** The difference quotients of a function $f(x)$ may be defined by the following recursion formulae

$$(4) \quad f(x; h_1) = \frac{f(x+h_1) - f(x)}{h_1}, \qquad (h_1 \neq 0),$$

$$f(x; h_1, \ldots, h_n) = \frac{f(x+h_n; h_1, \ldots, h_{n-1}) - f(x; h_1, \ldots, h_{n-1})}{h_n}, \qquad (n > 1).$$

It follows by a simple induction that

$$(5) \quad \begin{aligned} h_1 \cdots h_n f(x; h_1, \ldots, h_n) = {} & f(x + h_1 + \cdots + h_n) - \sum f(x + h_1 + \cdots + h_{n-1}) \\ & + \cdots + (-1)^r \sum f(x + h_1 + \cdots + h_{n-r}) + \cdots + (-1)^n f(x), \end{aligned}$$

where $\sum$ denotes the sum of all (algebraically) distinct terms which can be obtained by permuting $h_1, \ldots, h_n$ in the expression affected.

The formula which we seek to prove is

$$(6) \quad f(x; h_1, \ldots, h_n) = \sum f^{[n]}(x, x + h_1, x + h_1 + h_2, \ldots, x + h_1 + \cdots + h_n),$$

where $\sum$ has the same meaning as above. We prove rather an (in view of (5)) equivalent form of (6), viz.,

$$(6') \quad \sum_{i=1}^{n!} S_i\, h_1 \cdots h_n f^{[n]}(Z_0, \ldots, Z_n) = \frac{1}{n!} \sum_{i=1}^{n!} S_i \left\{ \sum_{r=0}^{n} (-1)^r \binom{n}{r} f(Z_{n-r}) \right\},$$

where Z_r denotes the polynomial $x + h_1 + \cdots + h_r$ $(r = 0, \ldots, n)$, $\binom{n}{r}$ denotes $n!/(n-r)!\, r!$, and $S_1, \ldots, S_{n!}$ are symbols for the $n!$ operations of replacing $h_1, \ldots, h_n$ by the $n!$ permutations of these quantities.

Now let S_j^i $(j = 1, \ldots, (n-1)!)$ denote those substitutions S_i such that h_n is carried into h_i and $h_1, \ldots, h_{n-1}$ are carried into the $(n-1)!$ permutations of the letters $h_1, \ldots, h_n$ other than h_i. Then if we assume that (6') has been proved for difference quotients of all orders lower than the nth, we have

$$\sum_{j=1}^{(n-1)!} S_j^i\, h_1 \cdots h_{n-1} f^{[n-1]}(Z_0, \ldots, Z_{n-1}) = \frac{1}{(n-1)!} \sum_{j=1}^{(n-1)!} S_j^i \left\{ \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} f(Z_{n-1-r}) \right\}, \quad (i = 1, \ldots, n),$$

so that on summing for $i = 1, \ldots, n$, we get

$$(7) \quad \sum_{i=1}^{n!} S_i\, h_1 \cdots h_{n-1} f^{[n-1]}(Z_0, \ldots, Z_{n-1}) = \frac{1}{(n-1)!} \sum_{i=1}^{n!} S_i \left\{ \sum_{r=0}^{n-1} (-1)^r \binom{n-1}{r} f(Z_{n-1-r}) \right\}.$$

Now let S_k^{ij} $(k = 1, \ldots, (n-2)!)$ denote the substitutions S_i which carry h_n and h_{n-1} into h_i and h_j respectively and $h_1, \ldots, h_{n-2}$ into the $(n-2)!$ permutations of $h_1, \ldots, h_n$ other than h_i and h_j. Then

$$\sum_{k=1}^{(n-2)!} S_k^{ij}\, h_1 \cdots h_{n-1} f^{[n-1]}(Z_0, \ldots, Z_{n-2}, Z_n)$$

$$= (n-2)! \sum_{k=1}^{n-2} S_k^{ij} h_{n-1} \left\{ \sum_{r=0}^{n-2} (-1)^r \binom{n-2}{r} f^{[1]}(Z_{n-2-r}, Z_n) \right\} \quad \left(\begin{matrix} (i, j = 1, \ldots, n) \\ i \neq j \end{matrix} \right),$$

so that

$$(8) \quad \sum_{i=1}^{n!} S_i\, h_1 \cdots h_{n-1} f^{[n-1]}(Z_0, \ldots, Z_{n-2}, Z_n) = \frac{1}{(n-2)!} \sum_{i=1}^{n!} S_i\, h_{n-1} \left\{ \sum_{r=0}^{n-2} (-1)^r \binom{n-2}{r} f^{[1]}(Z_{n-2-r}, Z_n) \right\}.$$

But

$$\sum_{i=1}^{n!} S_i\, h_{n-1}\, f^{[1]}(Z_{n-r}, Z_n) = \sum_{i=1}^{n!} S_i\, h_n\, f^{[1]}(Z_{n-r}, Z_n)$$

$$= \sum_{i=1}^{n!} S_i\, h_{n-2}\, f^{[1]}(Z_{n-r}, Z_n) = \cdots = \sum_{i=1}^{n!} S_i\, h_{n-r+1}\, f^{[1]}(Z_{n-r}, Z_n),$$

so that

$$\sum_{i=1}^{n!} S_i\, h_{n-1}\, f^{[1]}(Z_{n-r}, Z_n) = \sum_{i=1}^{n!} S_i \frac{1}{r}\,(h_{n-r+1} + \cdots + h_n)\, f^{[1]}(Z_{n-r}, Z_n)$$

$$= \sum_{i=1}^{n!} S_i \frac{1}{r}\,[f(Z_n) - f(Z_{n-r})] \qquad (r = 2, \ldots, n).$$

Hence (8) assumes the form

$$\sum_{i=1}^{n!} S_i\, h_1 \ldots h_{n-1}\, f^{[n-1]}(Z_0, \ldots, Z_{n-2}, Z_n)$$

(9)

$$= \frac{1}{(n-2)!} \sum_{i=1}^{n!} S_i \left\{ \sum_{r=0}^{n-2} (-1)^r \frac{1}{r+2} \binom{n-2}{r} [f(Z_n) - f(Z_{n-2-r})] \right\}.$$

Finally in view of the identity

$$h_n\, f^{[n]}(Z_0, \ldots, Z_n) = f^{[n-1]}(Z_0, \ldots, Z_{n-2}, Z_n) - f^{[n-1]}(Z_0, \ldots, Z_{n-2}, Z_{n-1})$$

and (7) and (8), we obtain

$$\sum_{i=1}^{n!} S_i\, h_1 \ldots h_n\, f^{[n]}(Z_0, \ldots, Z_n) = \frac{1}{n!} \sum_{i=1}^{n!} S_i \left\{ \sum_{r=0}^{n} A_{n-r} f(Z_{n-r}) \right\},$$

where

$$\frac{1}{n!} A_n = \frac{1}{(n-2)!} \left\{ \sum_{r=0}^{n-2} (-1)^r \binom{n-2}{r} \frac{1}{r+2} \right\} = \frac{1}{(n-2)!} \cdot \frac{1}{n-1} \cdot \frac{1}{n} = \frac{1}{n!},$$

$$\frac{1}{n!} A_{n-1} = -\frac{1}{(n-1)!} \binom{n-1}{0},$$

$$\frac{1}{n!} A_{n-2} = \frac{1}{(n-1)!} \binom{n-1}{1} - \frac{1}{2} \binom{n-2}{0} \frac{1}{(n-2)!},$$

$$\frac{1}{n!} A_{n-r} = (-1)^r \left\{ \binom{n-1}{r-1} \frac{1}{(n-1)!} - \frac{1}{r} \binom{n-2}{r-2} \frac{1}{(n-2)!} \right\} \quad (r = 3, \ldots, n);$$

from which it follows that

$$A_{n-r} = (-1)^r \binom{n}{r} \qquad (r = 0, \ldots, n).$$

This proves (6′), and hence (6).

3. **The definition of Ampere-Cauchy derived function extended.** Let us suppose that $x_0, \ldots, x_n$ are not all distinct but that $(i+1)$ of them are equal, say to a, and that the others are all distinct. We then define $f^{[n]}$ for this case by the equation

$$f^{[n]}(x_0, \ldots, x_n) = L_i \cdots L_1 \sum f^{[n]}(\xi_0, \ldots, \xi_n), \tag{10}$$

where L_k $(k = 1, \ldots, i)$ denotes the operation of taking the limit as h_k approaches 0 and $\xi_k = x_k + h_1 + \cdots + h_{m_k}$, m_k being the number of values less than k of r such that $x_r = a$, and where $0, h_1, h_1 + h_2, h_1 + \cdots + h_i$ and all their permutations are distinct. If $m_k = 0$, it is of course understood that $\xi_k = x_k$. Naturally the limit in the definition does not need to exist.

To extend the definition to cases where there are more than one set of equal values among the $x_0, \ldots, x_n$, we proceed by induction. Thus suppose that $f^{[n]}(x_0, \ldots, x_n)$ has been defined whenever there are less than s sets of equal values among the values $x_0, \ldots, x_n$. Then suppose there are s sets of equal values, the remaining values being distinct. Let $(j+1)$ denote the number of values in one set and b the common value. Then we define $f^{[n]}$ for this case by the equation

$$f^{[n]}(x_0, \ldots, x_n) = L_j \cdots L_1 \sum f^{[n]}(\eta_0, \ldots, \eta_n), \tag{11}$$

where L_k has the same meaning as before and $\eta_k = x_k + h_1 + \cdots + h_{l_k}$, l_k being the number of values less than k of r such that $x_n = b$, and where $0, h_1, h_1 + h_2, \ldots, h_1 + \cdots + h_j$ and all their permutations are distinct.

It is clear that $f^{[n]}$ as defined above is symmetric in its arguments, but it is not yet evident that its value does not depend on the order in which the definition is applied to the sets of equal values. That this order is immaterial will appear in the next section, where a symmetric formula for $f^{[n]}$ will be derived.

We note here that when all the arguments are equal, we get

$$f^{[n]}\left(\underbrace{x_0 \cdots x_0}^{n}\right) = \frac{1}{n!} f^{(n)}(x_0), \tag{12}$$

provided only that the derivative on the right exists.

4. **The development of $f^{[n]}$ in partial fractions.** If we for the moment set $F(x) = f^{[n-i]}(x x_{i+1} \cdots x_n)$ where $x, x_{i+1}, \ldots, x_n$ are supposed distinct, it is easily seen by aid of (12) that

$$f^{[n]}\left(\overset{0}{x_0} \cdots \overset{i}{x_0} x_{i+1} \cdots x_n\right) = \frac{F^{(i)}(x_0)}{i!}$$

if, and only if, the right hand member has a meaning. Thus applying (3) to $F(x)$ and differentiating i times, we get

$$f^{[n]}\left(\overset{0}{x_0} \cdots \overset{n}{x_0} x_{i-1} \cdots x_n\right) = \frac{1}{i!}\{D_x^i f(x)\, g(x)\}_{x=x_0}$$

$$+ \frac{f(x_{i+1})}{(x_{i+1}-x_0)^{i+1}(x_{i+1}-x_{i+2})\cdots(x_{i+1}-x_n)} + \cdots \tag{13}$$

$$\cdots + \frac{f(x_n)}{(x_n-x_0)^{i+1}(x_n-x_{i+1})\cdots(x_n-x_{n-1})},$$

where

$$g(x) = \frac{1}{(x-x_{i+1})\cdots(x-x_n)}.$$

Formula (13) certainly holds if $f^{(k)}(x_0)$ $(k = 1, \ldots, i)$ exist.

Formula (13) in expanded form is

$$f^{[n]}\left(\overset{0}{x_0} \cdots \overset{i}{x_0} x_{i+1} \cdots x_n\right) = \frac{1}{i!}\left\{\sum_{r=0}^{i}\binom{i}{r} f^{(i-r)}(x_0)\, g^{(r)}(x_0)\right\}$$

$$+ (x_{i-1}-x_0)^{i+1}(x_{i+1}-x_{i+2})\cdots(x_{i+1}-x_n) + \cdots \tag{14}$$

$$\cdots + (x_n-x_0)^{i+1}(x_n-x_{i+1})\cdots(x_n-x_{n-1}).$$

Formula (13) and (14) form the starting point for a proof by induction of the general formula

$$f^{[n]}\left(\overset{0}{x_0} \cdots \overset{i_0}{x_0} \overset{0}{x_1} \cdots \overset{i_1}{x_1} \cdots \overset{0}{x_k} \cdots \overset{i_k}{x_k}\right) = \sum_{r=0}^{k} \frac{1}{i_r!}\{D_x^{i_r} f(x)\, g_r(x)\}_{x=x_r}, \tag{15}$$

where

$$(16)\quad g_r(x) = \frac{1}{(x-x_0)^{i_0+1}\cdots(x-x_{r-1})^{i_{r-1}+1}(x-x_{r+1})^{i_{r+1}+1}\cdots(x-x_n)^{i_k+1}}.$$

Formula (15) certainly holds it $f^{(m_r)}(x_r)$ $(m_r = 1, \ldots, i_r)$ have a meaning, $(r = 0, \ldots, k)$.

5. **Lemmas on integration according to Duhamel and Serret.** Let $f(x)$ be a function defined on the closed interval $\overline{a\,b}$ and whose values may be either real numbers or $+\infty$ or $-\infty$. We then say that $\int_a^b f(x)\,dx$ exists and has the value I (in the sense of Duhamel and Serret) provided there is a function $F(x)$ continuous on $\overline{a\,b}$, differentiable at every point of $\overline{a\,b}$ and such that $F'(x) = f(x)$ on $\overline{a\,b}$ and $F(b) - F(a) = I$.

It is to be noted in connection with the above definition that the value of $\int_a^b f(x)\,dx$ is unique (if existent) provided that $f(x)$ has only finite values, otherwise, the value of the integral does not have to be unique.

By $\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f(x_1\cdots x_n)\,dx_1\cdots dx_n$ will be denoted the result (if existent) of operating on $f(x_1\cdots x_n)$ by $\int_{a_1}^{b_1} dx_1, \ldots, \int_{a_n}^{b_n} dx_n$ in the order written.

LEMMA 1. *If* $\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f(x_1\cdots x_n)\,dx_1\cdots dx_n$ *exists, then*

$$\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f(x_1\cdots x_n)\,dx_1\cdots dx_n = f(\xi_1\cdots\xi_n)(b_1-a_1)\cdots(b_n-a_n),$$

where

$$a_i < \xi_i < b_i, \qquad (i = 1, \ldots, n).$$

This lemma is equivalent for $n = 1$ to the mean value theorem in the calculus. The proof for any n is by induction, and will be omitted.

LEMMA 2. *If* $\int_{a_n}^{b_n}\cdots\int_{a_1}^{b_1} f(x_1\cdots x_n)\,dx_1\cdots dx_n$ *exists and has the value* I

and if $\varphi_1, \ldots, \varphi_n$ *are monotonic increasing functions on* $\overline{p_1 q_1}, \ldots, \overline{p_n q_n}$ *respectively such that* $\varphi_1', \ldots, \varphi_n'$ *exist and are finite on* $\overline{p_1 q_1}, \ldots, \overline{p_n q_n}$ *respectively and such that*

$$\varphi_1(p_1) = a_1, \ldots, \varphi_n(p_n) = a_n,$$

$$\varphi_1(q_1) = b_1, \ldots, \varphi_n(q_n) = b_n;$$

then $\int_{p_n}^{q_n} \cdots \int_{p_1}^{q_1} f[\varphi_1(t_1), \ldots, \varphi_n(t_n)]\, \varphi_1'(t_1) \cdots \varphi_n'(t_n)\, dt_1 \cdots dt_n$ *exists and has the value I.*

6. **Certain integral expressions for $f^{[n]}$.** Let us understand that all integrals written in this paper are in the sense of Duhamel and Serret.

Let a and b be respectively the smallest and the greatest values of $x_0, \ldots, x_n$. Then the formula

$$f^{[n]}(x_0, \ldots, x_n) = \int_0^1 f'^{[n-1]}[x_0 + (x_1 - x_0)t, \ldots, x_0 + (x_n - x_0)t]\, t^{n-1}\, dt \tag{17}$$

holds if $f, f', \ldots, f^{(m-1)}$ are continuous on $\overline{a\,b}$ and $f^{(m)}$ exists (finite or infinite) on $\overline{a\,b}$, where m is the greatest of the integers $i_0, i+1, \ldots, i_n+1$, where i_s is the number of times the value of x_s occurs in the set $x_0, \ldots, x_n$.

To prove (8) we write it in the more explicit form

$$f^{[n]}\left(\overset{0}{x_0} \cdots \overset{i_0}{x_0}\ \overset{0}{x_1} \cdots \overset{i_1}{x_1} \cdots \overset{0}{x_k} \cdots \overset{i_k}{x_k}\right) = \int_0^1 f'^{[n-1]}\left(\overset{0}{X_0} \cdots \overset{i_0-1}{X_0}\ \overset{0}{X_1} \cdots \overset{i_1}{X_1} \cdots \overset{0}{X_k} \cdots \overset{i_k}{X_k}\right) t^{n-1}\, dt, \tag{17'}$$

where

$$X_r = x_0 + (x_r - x_0)t, \qquad (r = 0, \ldots, k). \tag{18}$$

Now the equation

$$\sum_{r=0}^{k} \frac{1}{i_r!}\, g_r^{(i_r)}(x_r) = 0 \tag{19}$$

is what (15) becomes when $f(x) = 1$.

On applying (15) (expanded by Leibnitz theorem) to (17′), we see that (17′) is a consequence of (19) and the following set of equations

$$\frac{1}{i_0!}\{D^{i_0} f(x)\, g_0(x)\}_{x_0} = \int_0^1 \frac{1}{(i_0-1)!}\{D^{i_0-1} f'(x)\, G_0(x)\}_{X_0} t^{n-1}\, dt + \frac{1}{i_0!} f(x_0)\, g_0^{(i_0)}(x_0), \tag{20}$$

$$\frac{1}{i_r!}\{D^{i_r} f(x)\, g_r(x)\}_{x_r} = \int_0^1 \frac{1}{i_r!}\{D^{i_r} f'(x)\, H_r(x)\}_{X_r} t^{n-1}\, dt + \frac{1}{i_r!} f(x_0)\, g_r^{(i_r)}(x_0) \qquad (r = 1, \ldots, k) \tag{21}$$

where $G_r(x)$ is obtained from (16) by replacing $x_0, \ldots, x_k$ by $X_0, \ldots, X_k$ and where $h_r(x)$, $H_r(x)$ are obtained from $g_r(x)$, $G_r(x)$ respectively by replacing i_0 by $i_0 - 1$, that is,

$$h_r(x) = (x - x_0)\, g_r(x), \qquad H_r(x) = (x - x_0)\, G_r(x) \qquad (r = 1, \ldots, k). \tag{22}$$

From (22) we easily get

$$\begin{aligned} h_r^{(s)}(x) &= (x - x_0)\, g_r^{(s)}(x) + s\, g_r^{(s-1)}(x), \\ H_r^{(s)}(x) &= (x - x_0)\, G_r^{(s)}(x) + s\, G_r^{(s-1)}(x), \end{aligned} \qquad \begin{pmatrix} s = 1, 2, \ldots \\ r = 1, \ldots, k \end{pmatrix}, \tag{23}$$

from which follows

$$\binom{i_r}{s}(x - x_0)\, g_r^{(s)}(x) + (i_r - s + 1)\binom{i_r}{s-1} g_r^{(s-1)}(x) = \binom{i_r}{s} h_r^{(s)}(x), \qquad \begin{pmatrix} r = 1, \ldots, k \\ s = 1, \ldots, i_r \end{pmatrix} \tag{24}$$

since the left member is the sum of

$$\binom{i_r}{s}\left[(x - x_0)\, g_r^{(s)}(x) + s\, g_r^{(s-1)}(x)\right]$$

and the vanishing quantity

$$\left[(i_r - s + 1)\binom{i_r}{s} - s\binom{i_r}{s}\right] g_r^{(s-1)}(x).$$

Finally $g_r(x)$ is homogeneous of degree $i_r - n$ in the variables $x - x_0, \ldots, x - x_k$, so that $g_r^{(s)}$ is homogeneous of degree $i_r - n - s$ in those variables. Also $h_r^{(s)}$ is homogeneous of degree $i_r - n - s + 1$ in the same variables. Thus

$$(25) \qquad \begin{aligned} H_r^{(s)}(X) &= h_r^{(s)}(x)\, t^{i_r - n - s + 1}, \\ G_r^{(s)}(X) &= g_r^{(s)}(x)\, t^{i_r - n - s}. \end{aligned}$$

We now have

$$\frac{1}{i_0!}\{D^{i_0} f(x)\, g_0(x)\}_{X_0} = \frac{1}{i_0!}\sum_{s=0}^{i_0}\binom{i_0}{s} f^{(i_0-s)}(x_0)\, g_0^{(s)}(x_0).$$

Also

$$\begin{aligned} &\int_0^1 \frac{1}{(i_0-1)!}\{D^{i_0-1} f'(x)\, G_0(x)\}_{X_0}\, t^{n-1}\, dt \\ &= \int_0^1 \frac{1}{(i_0-1)!}\left\{\sum_{s=0}^{i_0-1}\binom{i_0-1}{s} f^{(i_0-s)}(x_0)\, G_0^{(s)}(x_0)\right\} t^{n-1}\, dt \\ &= \int_0^1 \frac{1}{(i_0-1)!}\left\{\sum_{s=0}^{i_0-1}\binom{i_0-1}{s} f^{(i_0-s)}(x_0)\, g_0^{(s)}(x_0)\, t^{i_0-s-1}\right\} dt \\ &= \frac{1}{(i_0-1)!}\left\{\sum_{s=0}^{i_0-1}\binom{i_0-1}{s}\frac{1}{i_0-s} f^{(i_0-s)}(x_0)\, g_0^{(s)}(x_0)\, t^{i_0-s}\right\}_0^1 \\ &= \frac{1}{i_0!}\left\{\sum_{s=0}^{i_0-1}\binom{i_0}{s} f^{(i_0-s)}(x_0)\, g_0^{(s)}(x_0)\, t^{i_0-s}\right\}_0^1 \\ &= \frac{1}{i_0!}\left\{\sum_{s=0}^{i_0-1}\binom{i_0}{s} f^{(i_0-s)}(x_0)\, g_0^{(s)}(x_0)\right\}. \end{aligned}$$

from which (20) follows.

Now consider (21) for $r = 1$. We have first

$$\begin{aligned} \frac{1}{i_1!}\{D^{i_1} f(x)\, g_1(x)\}_{x_1} &= \frac{1}{i_1!}\left\{\sum_{s=0}^{i_1}\binom{i_1}{s} f^{(i_1-s)}(x_1)\, g_1^{(s)}(x_1)\right\} \\ &= \frac{1}{i_1!}\left\{\sum_{s=0}^{i_1}\binom{i_1}{s} f^{(i_1-s)}(X_1)\, g_1^{(s)}(x_1)\, t^{i_1-s}\right\}_0^1 + \frac{1}{i_1!} f(x_0)\, g_1^{(i_1)}(x_1). \end{aligned}$$

Next

$$\int_0^1 \frac{1}{i_1!} \{D^{i_1} f'(x)\, H_1(x)\}_{X_1} t^{n-1}\, dt$$

$$= \frac{1}{i_1!} \int_0^1 \left\{ \sum_{s=0}^{i_1} \binom{i_1}{s} f^{(i_1+1-s)}(X_1)\, H_1^{(s)}(X_1) \right\} t^{n-1}\, dt$$

$$= \frac{1}{i_1!} \int_0^1 \left\{ \sum_{s=0}^{i_1} \binom{i_1}{s} f^{(i_1+1-s)}(X_1)\, h_1^{(s)}(x_1)\, t^{i_1-s} \right\} dt\,.$$

But it follows from (24) that

$$\frac{d}{dt} \left\{ \sum_{s=0}^{i_1} \binom{i_1}{s} f^{(i_1-s)}(X_1)\, g_1^{(s)}(x_1)\, t^{i_1-k} \right\} = \sum_{s=0}^{i_1} \binom{i_1}{s} f^{(i_1-s+1)}(X_1)\, h_1^{(s)}(x_1)\, t^{i_1-k},$$

from which (21) follows for $r = 1$. The proof of (21) for greater values of r is similar. The proof of (17) is thus complete.

If the more general hypothesis is made that $f, f', \ldots, f^{(n-1)}$ are continuous on $\overline{a\,b}$ and $f^{(m)}$ exists (finite or infinite) on $\overline{a\,b}$, where m is the greatest of the integers $i_0, i_1 + 1, \ldots, i_p + p, \ldots, i_k + p$ then (17) may be generalized to become

$$(22)\quad f^{[n]}(x_0, \ldots, x_n) = \int_0^1 \cdots \int_0^p f^{(p)[n-p]}(X_p^p \cdots X_n^p)\, t_1^{n-1} \cdots t_p^{n-p}\, dt_p \cdots dt_1$$

where

$$X_r' = X_r \qquad (r = 1, \ldots, n)$$

and X_r^p for $p > 1$ are defined by the recursion formula

$$(23)\quad X_r^p = X_{p-1}^{p-1} + (X_r^{p-1} - X_{p-1}^{p-1})\, t_p\,, \qquad \begin{pmatrix} p = 1, \ldots, n \\ r = p, \ldots, n \end{pmatrix}$$

The formula (22) is easily proved by induction as to p; the proof can be so formulated as to include the proof of the following formula

$$(24)\quad \begin{aligned} X_r^p &= x_0 + (x_1 - x_0)\, t_0 + (x_2 - x_1)\, t_1 t_2 + \cdots \\ &\cdots + (x_{p-1} - x_{p-2})\, t_1 \cdots t_{p-1} + (x_r - x_{p-1})\, t_1 \cdots t_p\,, \end{aligned} \qquad \begin{pmatrix} p = 1, \ldots, n \\ r = p, \ldots, n \end{pmatrix}$$

By means of the transformation $u_1 = t_1^n$, $u_2 = t_2^{n-1}$, $\ldots$, $u_p = t_p^{n+1-p}$, formula (22) may be written

$$(25)\quad f^{[n]}(x_0, \ldots, x_n) = \frac{(n-p)!}{n!} \int_0^{\overset{1}{\cdot 1}} \cdots \int_0^{\overset{p}{\cdot 1}} f^{(p)[n-p]}[X_p^p \cdots X_n^p]\, du_p \cdots du_1$$

where of course $X_p^p, \ldots, X_n^p$ are given by the equations (24) after transformation.

In the extreme case $p = n$, equation (22) reduces to

$$(26)\quad f^{[n]}(x_0, \ldots, x_n) = \int_0^{\overset{1}{\cdot 1}} \cdots \int_0^{\overset{n}{\cdot 1}} f^{(n)}[x_0 + (x_1 - x_0)\, t_1 + \cdots + (x_n - x_{n-1})\, t_1 \cdots t_n]\, t_1^{n-1} \cdots t_{n-1}\, dt_n \cdots dt_1,$$

and equation (25) reduces to

$$(27)\quad f^{[n]}(x_0, \ldots, x_n) = \frac{1}{n!} \int_0^{\overset{1}{\cdot 1}} \cdots \int_0^{\overset{n}{\cdot 1}} f^{(n)}\left[x_0 + (x_1 - x_0)\, u_1^{\frac{1}{n}} + \cdots + (x_n - x_{n-1})\, u_1^{\frac{1}{n}}\, u_2^{\frac{1}{n-1}} \cdots u_n\right] du_n \cdots du_1.$$

7. **Newton's interpolation formula with remainder term.** Consider the relation

$$(28)\quad f^{[n]}(x_0, \ldots, x_{n-1}, x) = f^{[n]}(x_0 \cdots x_{n-1}\, x_n) + (x - x_n)\, f^{[n+1]}(x_0 \cdots x_n x),$$

which holds for all distinct values of $x_0, \ldots, x_n, x$ by (2). Now, if $x = x_n$, the relation (28) is clearly true if the expressions involved have a meaning. In the same sense the relation holds if x_n is distinct from each of the quantities $x_0, \ldots, x_{n-1}$. The only case which needs attention is the case where x_n is equal to one or more of the $x_0, \ldots, x_{n-1}$. It will be sufficient to prove the relation

$$(29)\quad f^{[r]}\left(\overset{0}{x_0} \cdots \overset{r-1}{x_0}\, x\right) = f^{[r]}\left(\overset{0}{x_0} \cdots \overset{r}{x_0}\right) + (x - x_0)\, f^{[r+1]}\left(\overset{0}{x_0} \cdots \overset{r}{x_0} x\right),$$

since (28) for the case under consideration can always be reduced to this form by replacing f by a certain related function F, as is shown in § 4.

Now

$$f^{[r]}\left(\overset{0}{x_0}\cdots\overset{r-1}{x_0}\ x\right) = \frac{f(x)}{(x-x_0)^r} + \frac{1}{(r-1)!}\left\{D_t^{r-1}\frac{f(t)}{t-x}\right\}_{t=x_0}$$

$$= \frac{f(x)}{(x-x_0)^r} + \sum_{k=0}^{r-1}(-1)^k\frac{f^{(r-1-k)}(x_0)}{(r-1-k)!\,(x_0-x)^{k+1}}.$$

Therefore

$$f^{[r+1]}\left(\overset{0}{x_0}\cdots\overset{r}{x_0}\,x\right) = \frac{f(x)}{(x-x_0)^{r+1}} + \sum_{k=0}^{k}(-1)^r\frac{f^{(r-k)}(x_0)}{(r-k)!\,(x_0-x)^{k+1}}.$$

Also

$$f^{[r]}\left(\overset{0}{x_0}\cdots\overset{r}{x_0}\right) = \frac{f^{(r)}(x_0)}{r!}.$$

Substitution in (29) gives an identity, which proves (28).

The relation

$$\begin{aligned} f(x) = {} & f(x_0) + (x-x_0)\,f^{[1]}(x_0\,x_1) + (x-x_0)\,(x-x_1)\,f^{[2]}(x_0\,x_1\,x_2) \\ & + (x-x_0)\cdots(x-x_{n-1})\,f^{[n]}(x_0\cdots x_n) \\ & + (x-x_0)\cdots(x-x_n)\,f^{[n+1]}(x_0\cdots x_n\,x), \end{aligned} \tag{30}$$

is now easily proved by mathematical induction with the aid of (28); we omit the details.

Equation (30) may be written

$$f(x) = P(x) + R(x), \tag{31}$$

where

$$\begin{aligned} P(x) = {} & f(x_0) + (x-x_0)f^{[1]}(x_0\,x_1) + \cdots \\ & \cdots + (x-x_0)\cdots(x-x_{n-1})\,f^{[n]}(x_0\cdots x_n), \\ R(x) = {} & (x-x_0)\cdots(x-x_n)\,f^{[n+1]}(x_0\cdots x_n\,x). \end{aligned} \tag{32}$$

Consider the polynomial $P(x)$, which depends on the $(n+1)$ parameters $x_0,\ldots,x_n$. It is clear from (31) that $P(x)$ is symmetric in the quantities $x_0,\ldots,x_n$, since $R(x)$ is. Suppose r of the $x_0,\ldots,x_n$ are equal. On account of the symmetry we may suppose the notation so chosen that they are $x_0,\ldots,x_{r-1}$, that is $x_0 = x_1 = \cdots = x_{r-1}$. Then

$$\begin{aligned} P(x) = {} & f(x_0) + (x-x_0)\,f'(x_0) + \cdots + (x-x_0)^{r-1}\frac{f^{(r-1)}(x_0)}{(r-1)!} \\ & + (x-x_0)^r\,f^{[r]}\left(\overset{0}{x_0}\cdots\overset{r-1}{x_0}\ x_r\right) + \cdots \\ & + (x-x_0)^r\,(x-x_r)\cdots(x-x_{n-1})\,f^{[n]}\left(\overset{0}{x_0}\cdots\overset{r}{x_0}\,x_r\cdots x_n\right). \end{aligned}$$

On differentiating $P(x)$ k times ($k = 0, \ldots, r-1$) and setting $x = x_0$, we get

$$P^{(k)}(x_0) = f^{(k)}(x_0). \qquad (k = 0, \ldots, r-1).$$

We have thus the

THEOREM. *If among the numbers $x_0, \ldots, x_n$ there are r distinct ones $a_1 \cdots a_r$ repeated respectively $i_1 \cdots i_r$ times ($i_1 + \cdots + i_r = n+1$), then*

$$f^{(k)}(a_j) = P^{(k)}(a_j). \qquad (j = 1, \ldots, r;\ k = 0, \ldots, i_j - 1).$$

Let us now say that a is a root of $f(x)$ of multiplicity k (at least) if $f'(a), \ldots, f^{(k-1)}(a)$ exist and $f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0$. We may then say that $f(x) - P(x)$ has $(n+1)$ roots on $\overline{a\,b}$ if we count each root according to its multiplicity. Hence by Rolle's theorem, $f'(x) - P'(x)$ has there n roots, $f''(x) - P''(x)$ has there $(n-1)$ roots, $\ldots$, $f^{(n)}(x) - P^{(n)}(x)$ has there one root, say ξ. Thus

$$f^{(n)}(\xi) - n!\, f^{[n]}(x_0, \ldots, x_n) = 0$$

so that

$$f^{[n]}(x_0, \ldots, x_n) = \frac{f^{(n)}(\xi)}{n!}, \tag{33}$$

a formula valid if $f, f', \ldots, f^{(n-1)}$ are continuous on $\overline{a\,b}$ and $f^{(n)}$ exists (finite or infinite) on $\overline{a\,b}$.

Formula (33) also follows from (27) by aid of the lemmas of § 5.

Formula (33) gives

$$R(x) = (x - x_0) \cdots (x - x_n) \frac{f^{(n+1)}(\xi_x)}{(n+1)!}, \tag{34}$$

where ξ_x is between the smallest and the greatest of the numbers $x_0, \ldots, x_n, x$, and where there is an interval $\overline{a\,b}$ containing $x_0, \ldots, x_n, x$ such that $f, f', \ldots, f^{(n)}$ are continuous on $\overline{a\,b}$ and $f^{(n+1)}$ exists (finite of infinite) on $\overline{a\,b}$.

Various integral forms of $R(x)$ follow from (22).

MANILA, P. I.
November, 1922.

NOTE ON CERTAIN SEMINVARIANTS of n-LINES.

By Lennie Phoebe Copeland.

In this paper certain simultaneous seminvariants of pairs of m-lines, f_{3m}, g_{3m} are developed, the vanishing of which furnishes necessary and sufficient conditions in order that each form f_{3m}, g_{3m} should be expressible as a linear combination of the m m-th powers of the linear factors of the other. This theory is an extension of a known theorem on binary apolarity. There is also a relation between the subject of the present paper and a theory of pole-polygons (Pol-ecke) due to Rosanes and others.* When the above seminvariantive conditions are satisfied the two m-lines are designated as mutual pole-polygons.

Taking for the form of the general m-line†

$$f_{3m} = x_2^m l_{0x_1x_3} + x_2^{m-1} x_3 l_{1x_1x_3} + D^{-1}(-1)^{\frac{1}{2}m(m-1)} \sum_{j=2}^{m} x_3^j \sum_{i=0}^{m-j} \frac{\Delta_1^{m-j-i} \Delta_2^i R_m}{(m-j-i)!\, i!} x_1^{m-j-i} x_2^i = \prod_{i=1}^{m} (x_1 + r_i x_2 + s_i x_3),$$

where $r_i \neq r_j$, $s_i \neq s_j$, $r_i/s_i \neq r_j/s_j$, in which

$$\Delta_1 = m a_{00} \partial/\partial a_{10} + (m-1) a_{01} \partial/\partial a_{11} + \cdots + a_{0m-1} \partial/\partial a_{1m-1},$$

$$\Delta_2 = m a_{0m} \partial/\partial a_{1m-1} + (m-1) a_{0m-1} \partial/\partial a_{1m-2} + \cdots + a_{01} \partial/\partial a_{10},$$

$$x_2^{m-k} l_{kx_1x_3} = a_{k0} x_1^{m-k} + a_{k1} x_1^{m-k-1} x_2 + \cdots + a_{km-k} x_2^{m-k},$$

R is the resultant of $x_2^m l_{0x_1x_3}$ and $x_2^{m-1} l_{1x_1x_3}$, and D is the discriminant of $x_2^m l_{0x_1x_2}$, consider a pair of 2-lines,

* Copeland, On the Theory of invariants of n-lines, Annals of Math., vol. 16 (1914), p. 7. Grassmann, Gött. Nachrichten, December, 1872. London, Über die Polarfiguren der ebenen Curven dritter Ordnung, Math. Annalen, vol. 36 (1890), p. 535. Rosanes, Über ein Princip der Zuordnung algebraischer Formen, Crelle's Journal, vol. 76 (1873), p. 312.

† Glenn, The Theory of Invariants, Boston (1915), p. 202, and Trans. Amer. Math. Soc., vol. 12 (1911), p. 367.

$$f_{32} = a_{00}x_1^2 + a_{01}x_1x_2 + a_{02}x_2^2 + a_{10}x_1x_3 + a_{11}x_2x_3 - R_2/Dx_3^2$$
$$\equiv (x_1 + r_1x_2 + s_1x_3)(x_1 + r_2x_2 + s_2x_3) \equiv \alpha_{1x}\alpha_{2x} = 0,$$

$$g_{32} = b_{00}x_1^2 + b_{01}x_1x_2 + b_{02}x_2^2 + b_{10}x_1x_3 + b_{11}x_2x_3 - R_2'/D'x_3^2,$$
$$\equiv (x_1 + r_1'x_2 + s_1'x_3)(x_1 + r_2'x_2 + s_2'x_3) \equiv \beta_{1x}\beta_{2x} = 0, \quad (a_{00} = b_{00} = 1).$$

If we impose the condition that

$$\beta_{1x}\beta_{2x} = A_1\alpha_{1x}^2 + A_2\alpha_{2x}^2,$$

there result six equations in two unknowns giving the following matrix:

$$(1) \qquad \begin{vmatrix} 1 & 2r_1 & 2s_1 & r_1^2 & s_1^2 & 2r_1s_1 \\ 1 & 2r_2 & 2s_2 & r_2^2 & s_2^2 & 2r_2s_2 \\ 1 & \sum_1^2 r_i' & \sum_1^2 s_i' & r_1'r_2' & s_1's_2' & \sum_1^2 r_i's_j' \end{vmatrix} = 0,$$

which is equivalent to the four equations

$$(2) \qquad \begin{aligned} C_1 &= 2r_1r_2 - (r_1 + r_2)(r_1' + r_2') + 2r_1'r_2' = 0, \\ C_2 &= 2s_1s_2 - (s_1 + s_2)(s_1' + s_2') + 2s_1's_2' = 0, \\ C_3 &= 2r_1r_2s_1's_2' - (r_1s_2 + r_2s_1)(r_1's_2' + r_2's_1') + 2r_1'r_2's_1s_2 = 0, \\ C_4 &= (r_1' + r_2')(s_1 - s_2) + (s_1' + s_2')(r_2 - r_1) + 2(r_1s_2 - r_2s_1) = 0. \end{aligned}$$

Expressed in terms of the coefficients,

$$\begin{aligned} C_1 &= 2a_{00}b_{02} - a_{01}b_{01} + 2a_{02}b_{00} = 0, \\ C_2 &= 2a_{00}DR_2' + a_{10}b_{10}DD' + 2b_{00}D'R_2 = 0, \\ C_3 &= 2a_{02}DR_2' + a_{11}b_{11}DD' + 2b_{02}D'R_2 = 0, \\ C_4 &= a_{01}^2b_{10} - 4a_{02}b_{10} + 2a_{11}b_{01} - a_{01}a_{10}b_{01} + 4a_{02}a_{10} - 2a_{01}a_{11} = 0. \end{aligned}$$

Then $C_1 = 0$, $C_2 = 0$, $C_3 = 0$ are the three conditions for harmonic division on the sides of the triangle of reference,* and $C_4 = 0$ is one of two conditions relating to the positions of the vertices of f_{32}, g_{32}. A similar set of conditions C_1', C_2', C_3', C_4' exist if

$$\alpha_{1x}\alpha_{2x} = B_1\beta_{1x}^2 + B_2\beta_{2x}^2,$$

where $C_1' = C_1$, $C_2' = C_2$, $C_3' = C_3$, $C_4' = C_4(a \sim b)$ and $C_4' = 0$ is the second condition restricting the relative positions of the vertices.

This gives the theorem: *a set of necessary and sufficient conditions that f_{32}, g_{32} be mutual pole-polygons is given by the three conditions that f_{32}, g_{32} cut the three sides of the triangle of reference in apolar point-pairs, and the two conditions $C_4 = C_4' = 0$ relating to the vertices of f_{32}, g_{32}.*

Consider next the conditions that two triangles f_{33}, g_{33} be mutual pole-polygons. If we impose on the two 3-lines

$$f_{33} = \prod_{i=1}^{3}(x_1 + r_i x_2 + s_i x_3) \equiv \alpha_{1x}\alpha_{2x}\alpha_{3x} = 0,$$

$$g_{33} = \prod_{i=1}^{3}(x_1 + r_i' x_2 + s_i' x_3) \equiv \beta_{1x}\beta_{2x}\beta_{3x} = 0,$$

the condition that

$$\beta_{1x}\beta_{2x}\beta_{3x} = A_1\alpha_{1x}^3 + A_2\alpha_{2x}^3 + A_3\alpha_{3x}^3,$$

we have the matrix

$$(3)\qquad \begin{vmatrix} 1 & 3r_1 & 3s_1 & 3r_1^2 & 3s_1^2 & 6r_1s_1 & r_1^3 & s_1^3 & 3r_1^2s_1 & 3r_1s_1^2 \\ 1 & 3r_2 & 3s_2 & 3r_2^2 & 3s_2^2 & 6r_2s_2 & r_2^3 & s_2^3 & 3r_2^2s_2 & 3r_2s_2^2 \\ 1 & 3r_3 & 3s_3 & 3r_3^2 & 3s_3^2 & 6r_3s_3 & r_3^3 & s_3^3 & 3r_3^2s_3 & 3r_3s_3^2 \\ 1 & \sum_1^3 r_i' & \sum_1^3 s_i' & \sum_1^3 r_i'r_j' & \sum_1^3 s_i's_j' & \sum_1^3 r_i's_j' & r_1'r_2'r_3' & s_1's_2's_3' & \sum_1^3 r_i'r_j's_k' & \sum_1^3 r_i's_j's_k' \end{vmatrix} = 0,$$

$$(i \neq j \neq k),$$

* Grace and Young, Algebra of Invariants, Cambridge, (1903), p. 213.

which is equivalent to the seven equations

$$C_1 = 3\,r_1\,r_2\,r_3 - \sum_1^3 r_i\,r_j \sum_1^3 r'_i + \sum_1^3 r_i \sum_1^3 r'_i\,r'_j - 3\,r'_1\,r'_2\,r'_3 = 0,$$

$$C_2 = 3\,s_1\,s_2\,s_3 - \sum_1^3 s_i\,s_j \sum_1^3 s'_i + \sum_1^3 s_i \sum_1^3 s'_i\,s'_j - 3\,s'_1\,s'_2\,s'_3 = 0,$$

$$C_3 = 3\,r_1\,r_2\,r_3\,s'_1\,s'_2\,s'_3 - \sum_1^3 r_i\,r_j\,s_k \sum_1^3 r'_i\,s'_j\,s'_k + \sum_1^3 r_i\,s_j\,s_k \sum_1^3 r'_i\,r'_j\,s'_k - 3\,r'_1\,r'_2\,r'_3\,s_1\,s_2\,s_3 = 0,$$

$$C_4 = U r_1^2 + V r_2^2 + W r_3^2 + Y \sum_1^3 r'_i\,r'_j = 0,$$

$$C_5 = 2\,U r_1\,s_1 + 2\,V r_2\,s_2 + 2\,W r_3\,s_3 + Y \sum_1^3 r'_i\,s'_j = 0,$$

$$C_6 = U r_1^2\,s_1 + V r_2^2\,s_2 + W r_3^2\,s_3 + Y \sum_1^3 r'_i\,r'_j\,s'_k = 0,$$

$$C_7 = U r_1\,s_1^2 + V r_2\,s_2^2 + W r_3\,s_3^2 + Y \sum_1^3 r'_i\,s'_j\,s'_k = 0,$$

where

$$\begin{aligned} U &= -\sum_1^3 s'_i\,(r_3 - r_2) + \sum_1^3 r'_i\,(s_3 - s_2) - 3\,(r_2\,s_3 - r_3\,s_2), \\ V &= \sum_1^3 s'_i\,(r_3 - r_1) - \sum_1^3 r'_i\,(s_3 - s_1) + 3\,(r_1\,s_3 - r_3\,s_1), \\ W &= -\sum_1^3 s'_i\,(r_2 - r_1) + \sum_1^3 r'_i\,(s_2 - s_1) - 3\,(r_1\,s_2 - r_2\,s_1), \\ Y &= s_3\,(r_2 - r_1) - r_3\,(s_2 - s_1) + (r_1\,s_2 - r_2\,s_1). \end{aligned} \tag{4}$$

The expressions C_1, C_2, C_3 give the conditions that f_{33} and g_{33} cut the sides of the triangle of reference in apolar ranges. Expressed in terms of the coefficients they are as follows:*

$$C_1 = 3\,a_{00}\,b_{03} - a_{01}\,b_{02} + a_{02}\,b_{01} - 3\,a_{03}\,b_{00} = 0,$$

$$C_2 = 3\,a_{00}\,D\,R'_3 - a_{10}\,D\,\Delta_1\,R'_3 + b_{10}\,D'\,\Delta_1\,R_3 - 3\,b_{00}\,D'\,R_3 = 0,$$

$$C_3 = 3\,a_{03}\,D\,R'_3 - a_{12}\,D\,\Delta_2\,R'_3 + b_{12}\,D'\,\Delta_2\,R_3 - 3\,b_{03}\,D'\,R_3 = 0;$$

* Cf. Copeland, Loc. cit.

and U, V, W, Y are equivalent to four possible conditions (C_4), (cf. (2)), upon the vertices of f_{33}, g_{33}. A similar set of seminvariantive expressions $C_1', C_2', C_3', C_4', C_5', C_6', C_7'$, give the condition that

$$\alpha_{1x}\,\alpha_{2x}\,\alpha_{3x} = B_1\,\beta_{1x}^3 + B_2\,\beta_{2x}^3 + B_3\,\beta_{3x}^3,$$

where $C_1' = C_1$, $C_2' = C_2$, $C_3' = C_3$, and

$$C_4' = U'\,r_1'^2 + V'\,r_2'^2 + W'\,r_3'^2 + Y'\sum_1^3 r_i\,r_j = 0,$$

$$C_5' = 2\,U'\,r_1'\,s_1' + 2\,V'\,r_2'\,s_2' + 2\,W'\,r_3'\,s_3' + Y'\sum_1^3 r_i\,s_j = 0,$$

$$C_6' = U'\,r_1'^2\,s_1' + V'\,r_2'^2\,s_2' + W'\,r_3'^2\,s_3' + Y'\sum_1^3 r_i\,r_j\,s_k = 0,$$

$$C_7 = U'\,r_1'\,s_1'^2 + V'\,r_2'\,s_2'^2 + W'\,r_3'\,s_3'^2 + Y'\sum_1^3 r_i\,s_j\,s_k = 0,$$

where U', V', W', Y' are obtained from U, V, W, Y respectively by interchanging the primed and unprimed sets.

The preceding theory can be generalized for a similar set of necessary and sufficient conditions that f_{3m}, g_{3m} be mutual pole-polygons.

WELLESLEY,
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THE FREDHOLM THEORY OF STIELTJES INTEGRAL EQUATIONS.

By C. A. Fischer.

F. Riesz has derived a large part of the Fredholm theory for an equation involving a completely continuous transformation,* which is equivalent to the Stieltjes integral equation

$$f(x) = \varphi(x) - \lambda \int_a^b \varphi(y)\, d_y\, K(x, y), \tag{1}$$

in which $K(x, y)$ must have certain properties,† without the use of the Fredholm determinant.

In the present paper the Fredholm determinant for this equation is defined in terms of modified Stieltjes integrals, and the classical Fredholm theory is applied directly. In the first section some properties of Young-Stieltjes integrals are obtained, including a theorem on reversing the order of iterated integrals. Young-Stieltjes integrals are used throughout the paper. In § 2 the Fredholm determinant is defined and proved to be determined uniquely when $K(x, y)$ satisfies certain conditions, and the solution of equation (1) is found. In the next section the homogeneous equation is discussed briefly, and in the last section it is proved that this theory does not include, and is not included by, the Riesz theory.

1. **Some properties of Young-Stieltjes integrals.** The set $\{f(x)\}$ will consist of all bounded, real valued functions defined on (a, b), for which the Young-Stieltjes integral exists, with respect to a function of finite variation.‡

As all the integrals considered in this paper will be definite integrals with the limits a and b, these limits will be omitted in the notation, and also $d_y K(x, y)$ will be written simply $d K(x, y)$. Since the integral in equation (1) is unchanged when $K(x, y) - K(x, a)$ is substituted for $K(x, y)$, it can be assumed that $K(x, a) = 0$. It can also be assumed that

$$K(x, y) = K(x, y + 0), \qquad (a < y < b). \tag{2}$$

* F. Riesz, Acta Mathematica, vol. 41, (1918), pp. 71–98.

† Fischer, Bulletin American Mathematical Society, vol. 27, (1920), p. 14.

‡ Young, Proc. Lond. Math. Soc., vol. 13, (1914), p. 128, or Daniell, Annals of Math., vol. 19, (1918), pp. 287–8.

without affecting the same integral. Both these assumptions will be made throughout the paper.

If $f(x)$ belongs to $\{f(x)\}$ and $\alpha(x)$ is monotone increasing, there must be a sequence of upper semi-continuous functions $g_1(x), g_2(x), \ldots$, and a sequence of lower semi-continuous functions $h_1(x), h_2(x), \ldots$, such that $g_n(x) \leq f(x)$, $h_n(x) \geq f(x)$, and the equations

$$\lim_{n\to\infty} \int g_n(x)\, d\alpha(x) = \lim_{n\to\infty} \int h_n(x)\, d\alpha(x) = \int f(x)\, d\alpha(x)$$

are satisfied.* The sequence of g's for any given x can be made monotone increasing with respect to n by substituting the largest of $g_1(x), g_2(x), \ldots$ $\ldots . g_n(x)$ for $g_n(x)$, and the sequence of h's monotone decreasing in a similar way. It has also been proved that, if each function of a bounded sequence $f_1(x), f_2(x), \ldots .$ approaching $f(x)$, belongs to $\{f(x)\}$, then $f(x)$ also belongs to the same set, and $\int f_n(x)\, d\alpha(x)$ approaches $\int f(x)\, d\alpha(x)$.†

The variation of $K(x, y)$ in y will be designated by $V_y K(x, y)$, or $V_y K(x, y)\Big|_{y_0}^{y_2}$ if the interval considered is (y_1, y_2) instead of (a, b).

If $K(x, y)$ belongs to $\{f(x)\}$ for every constant value of y, if $V_y K(x, y)$ is bounded uniformly and if $\varphi(x)$ is continuous, the integral

$$\int \varphi(y)\, d_y K(x, y) \tag{3}$$

is approached by a sequence of functions of x such as

$$\sum_{i=1}^{n} \varphi(\eta_i)\, [K(x, y_i) - K(x, y_{i-1})], \qquad (y_{i-1} \leq \eta_i \leq y_i),$$

each of which belongs to $\{f(x)\}$, and therefore the integral (3) belongs to that set also. If $\varphi(x)$ belongs to $\{f(x)\}$ but is not continuous, the integral (3) can be approached by a sequence of similar integrals for semi-continuous φ's, each of which is approached by a similar sequence for continuous φ's. Consequently the integral (3) must belong to $\{f(x)\}$ in this case also. Similarly it can be proved that if $K(x, y)$ satisfies the same conditions $V_y K(x, y)\Big|_a^y$ must also belong to $\{f(x)\}$ for every value of y.

* Young or Daniell, loc. cit.

† Daniell, loc. cit., p. 290.

To simplify the notation $\|f(x)\|$ will represent the constant least upper bound of $|f(x)|$, and similarly $\|a_n\|$ will represent that of $|a_n|$, where $a_1, a_2, \ldots$, may be either a finite or an infinite sequence of numbers.

In order to justify reversing the order of integration of an iterated Young-Stieltjes integral, two theorems proved by Bray must be extended.*

The first was that, if $V\gamma(x)$ is finite, $V_y K(x, y)$ bounded uniformly, and $K(x, y)$ continuous in x for every value of y, then the function

$$\psi(y) = \int K(x, y)\, d\gamma(x) \tag{4}$$

has finite variation. The assumption that $K(x, y)$ be continuous in x will now be replaced by the assumption that it simply belongs to $\{f(x)\}$ for every value of y.

If $\psi(y)$ is put into the form

$$\psi(y) = \int [K_1(x, y) - K_2(x, y)]\, d[\gamma_1(x) - \gamma_2(x)].$$

where $K_1(x, y) = V_y K(x, y)\Big|_a^y$ and $K_2(x, y) = K(x, y) - K_1(x, y)$, and $\gamma_1(x)$ and $\gamma_2(x)$ are defined similarly, $K_1(x, y)$ and $K_2(x, y)$ will belong to $\{f(x)\}$ for every constant value of y and will be monotone increasing in y, and $\gamma_1(x)$ and $\gamma_2(x)$ will also be monotone increasing. The functions $K_1(x, y)$ and $K_2(x, y)$ can be readily proved to satisfy equation (2). It will then be sufficient to prove the theorem for the case where $K(x, y)$ and $\gamma(x)$ are monotone increasing in the proper arguments.

It follows from Bray's proof of the theorem that when $K(x, y)$ is continuous in x,

$$V_y \psi(y) \leqq 2 \| V_y K(x, y) \| V\gamma(x).$$

This inequality, and the fact that the variation of the limit of a sequence of functions cannot be greater than an upper bound of the variations of the functions, prove that $V\psi$ will be finite if the variations of all the K's in sequences used in defining the integral (4) for a discontinuous $K(x, y)$, have a common upper bound.

Since $K(x, y)$ belongs to $\{f(x)\}$ for every y, there must be a monotone increasing sequence of functions $\{K_n(x, y)\}$, each of which is upper semi-

* Bray, Annals of Math., vol. 20 (1919), pp. 181–5.

continuous in x and less than or equal to $K(x, y)$, which satisfy the equation

$$\lim_{n=\infty} \int K_n(x, y)\, d\gamma(x) = \int K(x, y)\, d\gamma(x).$$

Since $K(x, a) = 0$, it is possible to make $K_n(x, a) = 0$ for all values of x and n.

A denumerable set of points $\{y_j\}$ will now be selected which is dense on (a, b) and includes both end points of the interval. If the value of $K(x, y)$ is given at these points, equation (2) determines its value elsewhere. A new sequence of functions will be defined by the equations

$$H_n(x, y) = \text{max. } K_n(x, y_j), \quad (j \leqq n;\ y_j \leqq y;\ n = 1, 2, \ldots).$$

It follows directly that $H_n(x, y)$ is monotone increasing in y, upper semi-continuous in x, not greater than $K(x, y)$, and satisfies the equations

$$\lim_{n=\infty} \int H_n(x, y_j)\, d\gamma(x) = \int K(x, y_j)\, d\gamma(x), \quad (j = 1, 2, \ldots). \tag{5}$$

Since $H_1(x, y)$ is upper semi-continuous in x, it can be approached by a monotone decreasing sequence $H_{11}(x, y)$, $H_{12}(x, y), \ldots$ of functions which are continuous in x. Since $H_2(x, y) \geqq H_1(x, y)$, and the greatest of a finite number of continuous functions is continuous, the similar sequence approaching $H_2(x, y)$ can be made to satisfy the inequalities $H_{2r} \geqq H_{1r}$. In this way a doubly infinite sequence of functions can be defined, each function continuous in x, which satisfies the inequalities

$$\begin{array}{ccccccc}
H_{11}(x, y) & \geqq & H_{12}(x, y) & \geqq & \cdots & \longrightarrow & H_1(x, y) \\
\| \wedge & & \| \wedge & & & & \| \wedge \\
H_{21}(x, y) & \geqq & H_{22}(x, y) & \geqq & \cdots & \longrightarrow & H_2(x, y) \\
\| \wedge & & \| \wedge & & & & \| \wedge \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array} \tag{6}$$

If $\|K\|$ is substituted for $H_{nr}(x, y)$ wherever $H_{nr}(x, y) > \|K\|$, the set $\{K_{nr}(x, y)\}$ will be bounded uniformly. The new functions $G_{nr}(x, y)$ will now be defined by the equations

$$G_{nr}(x, y) = \text{min. } H_{nr}(x, y_j), \quad (j \leqq r;\ y_j \geqq y;\ r, n = 1, 2, \ldots).$$

These functions are continuous in x, satisfy inequalities (6) with G_{nr} substituted for H_{nr} and new functions G_n substituted for $H_n(x, y)$, and are also

monotone increasing in y. Since $H_n(x, y)$ is monotone increasing in y, it follows from definition that $G_{nr}(x, y)$, and consequently $G_n(x, y)$ cannot be less than $H_n(x, y)$, and that

$$(7) \quad G_n(x, y_j) = H_n(x, y_j), \quad (a \leq x < b;\ j = 1, 2, \ldots, n;\ n = 1, 2, \ldots).$$

If the limit of $G_n(x, y)$ as n becomes infinite, is called $G(x, y)$, equations (5) and (7) imply that

$$(8) \qquad \int G(x, y_j)\, d\gamma(x) = \int K(x, y_j)\, d\gamma(x) = \psi(y_j), \qquad (j = 1, 2, \ldots).$$

Since the $G_{nr}(x, y)$ have uniformly bounded variation in y, the expression

$$(9) \qquad V_y \int G(x, y)\, d\gamma(x)$$

is finite.

But equations (8) and (2), and the fact that the y_j's are dense on (a, b) and include a and b, imply that $V\psi(y)$ cannot be greater than (9), and is therefore finite.

The second theorem is that if $f(y)$ is continuous, $K(x, y)$ continuous in x, $V_y K(x, y)$ bounded uniformly, and $V\gamma(x)$ is finite, the following iterated integrals exist and satisfy the equation

$$(10) \qquad \int \left[\int f(y)\, d\, K(x, y) \right] d\gamma(x) = \int f(y)\, d_y \int K(x, y)\, d\gamma(x).$$

In this theorem the hypothesis that $f(y)$ be continuous and $K(x, y)$ continuous in x, will be replaced by the assumption that they both belong to $\{f(x)\}$. It follows from the previous theorem that the iterated integral in the right member must exist, and since it has been proved that $\int f(y)\, d\, K(x, y)$ must belong to $\{f(x)\}$, the integral on the left must exist also. If f is continuous, but $K(x, y)$ not necessarily so, and the points $y_0 = a < y_1 < y_2 < \cdots$ $\cdots < y_n = b$, are taken so near together that the oscillation of $f(y)$ in every interval of length $y_i - y_{i-1}$ is less than ϵ, the absolute value of the difference between the left member of equation (10) and the expression

$$\sum_1^n f(y_i) \left[\int K(x, y_i)\, d\gamma(x) - \int K(x, y_{i-1})\, d\gamma(x) \right]$$

cannot be greater than $\varepsilon \| V_y K \| V\gamma$,* and the absolute value of the difference between the right member and the same expression cannot be greater than $\varepsilon V\psi$. Since both these expressions approach zero with ε, equation (10) must be satisfied.

The condition that f be continuous will now be removed. As every semi-continuous function is the limit of a bounded sequence of continuous functions, equation (10) can be immediately extended to the case where f is semi-continuous.† Since $\gamma(x)$ and $K(x, y)$ are monotone increasing in x and y respectively, $\psi(y)$ must be monotone increasing. If f is any function belonging to $\{f(x)\}$, the upper semi-continuous functions $g_n(y) \leqq f(y)$ which satisfy the equation

$$\lim_{n\to\infty} \int g_n(y)\, d\psi(y) = \int f(y)\, d\psi(y),$$

must also satisfy

$$\int \left[\int f(y)\, d K(x, y) \right] d\gamma(x) \geqq \int \left[\int g_n(y)\, d K(x, y) \right] d\gamma(x) = \int g_n(y)\, d\psi(y), \qquad (n = 1, 2, \ldots). \tag{11}$$

Also the lower semi-continuous functions $h_n(y) \geqq f(y)$ which satisfy the equations

$$\lim_{n\to\infty} \int h_n(y)\, d\psi(y) = \int f(y)\, d\psi(y),$$

must also satisfy

$$\int \left[\int f(y)\, d K(x, y) \right] d\gamma(x) \leqq \int \left[\int h_n(y)\, d K(x, y) \right] d\gamma(x) = \int h_n(y)\, d\psi(y), \qquad (n = 1, 2, \ldots). \tag{12}$$

But the integrals in the right members of equations (11) and (12) both approach $\int f(y)\, d_y \int K(x, y)\, d\gamma(x)$ as n becomes infinite. Consequently equation (10) must be satisfied in this case also.

* Bray, loc. cit., p. 179.

† Daniell, loc. cit., p. 290.

2. **The solution of equation** (1). In this section the Fredholm determinant for equation (1) will be defined and conditions found under which the classical Fredholm theory* can be applied with very little modification.

Equation (1) may be regarded as the limit of a sequence of systems of equations such as

$$f(x_i) = \varphi(x_i) - \lambda \sum_{j=1}^{n} \varphi(y_j)\, \Delta K(x_i, y_j), \qquad (j = 1, 2, \ldots, n),$$

where

$$\Delta K(x_i, y_j) = K(x_i, y_j) - K(x_i, y_{j-1}).$$

The determinant of this system can be reduced to the form

$$D_n(\lambda) = 1 - \lambda \sum_{i=1}^{n} \Delta K(x_i, y_i) + \frac{\lambda^2}{2!} \sum_{i,j=1}^{n} \begin{vmatrix} \Delta K(x_i, x_i)\, \Delta K(x_i, x_j) \\ \Delta K(x_j, x_i)\, \Delta K(x_j, x_j) \end{vmatrix} - \cdots$$

(13)

$$+ (-1)^n \lambda^n \begin{vmatrix} \Delta K(x_1, x_1)\, \Delta K(x_1, x_2) \cdots \Delta K(x_1, x_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \Delta K(x_n, x_1)\, \Delta K(x_n, x_2) \cdots \Delta K(x_n, x_n) \end{vmatrix}.$$

It will be assumed that the expression

$$V_2 K, = \text{least upper bound} \sum_{i=1}^{n} |\Delta K(x_i, y_i)|,$$

is finite, where $x_1, x_2, \ldots, x_n$ is an arbitrary set of points on (a, b), and $y_1, y_2, \ldots, y_n$ is also arbitrary excepting that $a = y_0 < y_1 < \cdots < y_n = b$. It then follows from Hadamard's theorem that the general term of $D_n(\lambda)$ satisfies the inequality

$$\left| \frac{\lambda^m}{m!} \sum_{i_1, i_2, \ldots, i_m = 1}^{n} \begin{vmatrix} \Delta K(x_{i_\alpha}, x_{i_\beta}) \\ (\alpha, \beta = 1, 2, \ldots, m) \end{vmatrix} \right| \leq \frac{m^{\frac{m}{2}} |\lambda|^m}{m!} (V_2 K)^m.$$

Consequently $D_n(\lambda)$ is bounded uniformly in n, for every finite λ, and if each term approaches a unique limit as n becomes infinite, the resulting series will be absolutely convergent. This series, which will be represented by the notation

* See for instance, Bôcher, Introduction to Integral Equations, pp. 24-43.

$$(14)\quad D(\lambda) = 1 - \lambda \int dK(\xi,\xi) + \frac{\lambda^2}{2!}\int\int \begin{vmatrix} dK(\xi,\xi) & dK(\xi,\eta) \\ dK(\eta,\xi) & dK(\eta,\eta) \end{vmatrix} - \cdots$$

constitutes the Fredholm determinant for the problem.

This series involves the two types of integrals

$$(15)\quad \int dK(\xi,\xi) = \lim_{\|\Delta\xi_i\|\to 0} \sum_{i=1}^{m} \Delta K(\xi_i,\xi_i),$$

and

$$\int\int\int dK(\xi,\eta)\, dK(\eta,\zeta)\, dK(\zeta,\xi)$$

$$(16)\quad = \lim_{\|\Delta\xi_i\|\text{etc.}\to 0} \sum_{i,j,k=1}^{m,n,p} \Delta K(\xi_i,\eta_j)\, \Delta K(\eta_j,\zeta_k)\, \Delta K(\zeta_k,\xi_i).$$

It will be understood that $a = \xi_0 < \xi_1 < \cdots < \xi_m = b$ etc., but not necessarily that $m = n = p$, not that $\xi_1 = \eta_1$ etc.

The conditions under which these integrals are determined uniquely will involve the expression

$$V_\delta K(x,y)\Big|_R = \text{least upper bound} \sum_{i=1}^{n} |\Delta K(\xi_i',\eta_i) - \Delta K(\xi_i,\eta_i)|,$$

where (ξ_i,η_i) and (ξ_i',η_i) are arbitrary points of the region R, excepting that $a = \eta_0 < \eta_1 < \cdots < \eta_n = b$, and $|\xi_i' - \xi_i| < \delta$.

It will then be assumed that there is a sequence $x_1, x_2, \ldots$, which has the property that, when an $\varepsilon > 0$ is chosen there must be a finite r such that, after another positive constant $\varepsilon' > 0$ is taken, there will always be a $\delta > 0$ which satisfies the inequality

$$(17)\quad V_\delta K(x,y)\Big|_{R'} < \varepsilon,$$

where R' is all of the region $(a \leqq x \leqq b;\ a \leqq y \leqq b)$, excepting the strips where the inequalities $(\eta_\alpha - \varepsilon' \leqq x \leqq \eta_\alpha + \varepsilon',\ \alpha = 1, 2, \ldots, r)$ are satisfied. For convenience δ will always be chosen less than ε'. If $K_{xy}(x,y)$ is bounded uniformly in R', the mean value theorem implies that $V_\delta K(x,y)\Big|_{R'} \leqq \delta(b-a)\,|K_{xy}(x,y)|$.

It will further be assumed that

$$(18)\quad \lim_{\varepsilon\to 0} V_\varepsilon K(x,y)\Big|_{\eta_\alpha - \varepsilon}^{\eta_\alpha + \varepsilon} = 0, \qquad (\alpha = 1, 2, \ldots).$$

These conditions will always be satisfied if $\left|\frac{\partial K(x, y)}{\partial y}\right|$ and $\left|\frac{\partial^2 K(x, y)}{\partial x \partial y}\right|$ are bounded, but this is not necessary as $K(x, y)$ need not be continuous in x at the points $x = x_\alpha$, and may be discontinuous in y when $y \neq x_\alpha$.

To prove the existence of the integral (15), an $\epsilon > 0$ will be taken arbitrarily small. Then it will be proved that there is a $\delta > 0$ such that, if S is the sum in the right member of equation (15) corresponding to a partition such that $|\Delta \xi_i| < \delta$ and S' the same sum after the intervals (ξ_{i-1}, ξ_i) are further subdivided, the inequality $S' - S < 2\epsilon$ will be satisfied. If this is the case and S'' is a similar sum for which $|\Delta \xi_i| < \delta$, but which does not involve all the ξ_i's which where used in S, S' can be chosen so that it will involve every ξ_i which was used in either S or S'', and the inequality

$$|S'' - S| \leq |S' - S| + |S' - S''| < 4\epsilon$$

will be satisfied. Then the integral (15) must be determined uniquely.

After ϵ has been chosen and the corresponding r determined, ϵ' will be taken so small that

$$V_2 K(x, y)\Big|_{\eta_\alpha - 2\epsilon'}^{\eta_\alpha + 2\epsilon'} < \frac{\epsilon}{2r}, \qquad (\alpha = 1, 2, \ldots, r). \tag{19}$$

Then δ will be determined so as to satisfy inequality (17), and S and S' will be taken as explained above. The ξ's corresponding to S' will be designated by $\xi'_0, \xi'_1, \ldots, \xi'_{n'}$. Then every ξ_i will be a $\xi'_{i'}$ but not vice versa. This makes it possible to reduce $S' - S$ to the form

$$S' - S = \sum_{i'=1}^{n'} [\Delta K(\xi'_{i'}, \xi'_{i'}) - \Delta K(\xi_i, \xi'_{i'})] \tag{20}$$

where i varies with i' in such a way that $\xi_{i-1} \leq \xi'_{i'-1} < \xi'_{i'} \leq \xi_i$. U will now represent the terms in (20) such that ξ_{i-1} and ξ_i are both in some interval $(\eta_\alpha - 2\epsilon', \eta_\alpha + 2\epsilon')$ with $\alpha \leq r$, and V the other terms. Since $\delta < \epsilon'$, no ξ_i nor $\xi'_{i'}$ involved in V can be in one of the r intervals $(x_\alpha - \epsilon', x_\alpha + \epsilon')$. Inequality (19) implies that

$$|U| \leq 2 \sum_{\alpha=1}^{r} V_2 K(x, y)\Big|_{\eta_\alpha - 2\epsilon'}^{\eta_\alpha + 2\epsilon'} < \epsilon,$$

and inequality (17) that

$$|V| \leq V_\delta K\Big|_{R_{\epsilon'}} < \epsilon.$$

Thus $|S' - S| < 2\epsilon$, and the integral (15) is determined uniquely.

The triple integral (16) will now be considered. After ϵ, r, ϵ' and δ are chosen as before, S will represent the sum in the right member of equation (16) corresponding to a partition such that $\Delta\xi_i$, $\Delta\eta_j$ and $\Delta\zeta_k$ are all less than δ, and S' the same sum after the intervals (ξ_{i-1}, ξ_i) have been further divided, without changing any of the intervals (η_{j-1}, η_j) or (ζ_{k-1}, ζ_k). Proceeding as before,

$$S' - S = \sum_{i',j,k=1}^{m',n,p} [\Delta K(\xi'_{i'}, \eta_j) - \Delta K(\xi_i, \eta_j)]\, \Delta K(\eta_j, \zeta_k)\, \Delta K(\zeta_k, \xi'_{i'}).$$

Then if U represents the sum of the terms such that ξ_{i-1} and ξ_i are in the same interval $(x_\alpha - 2\epsilon', \eta_\alpha + 2\epsilon')$ with $\alpha \leq r$, and V the other terms, the inequalities

$$|U| \leq 2(V_2 K)^2 \sum_{\alpha=1}^{r} V_2 K(x, y)\Big|_{\eta_\alpha - 2\epsilon'}^{\eta_\alpha + 2\epsilon'} < \epsilon (V_2 K)^2,$$

and

$$|V| \leq (V_2 K)^2 V_\delta K\Big|_{R_{\epsilon'}} < \epsilon (V_2 K)^2$$

will be satisfied. Thus $|S' - S| < 2\epsilon (V_2 K)^2$. If the intervals (η_{j-1}, η_j) and (ζ_{k-1}, ζ_k) had also been subdivided, this inequality would have been $S' - S| < 6(V_2 K)^2$, which also approaches zero with ϵ. The proof of the existence of an n-tuple integral similar to (16) offers no new difficulties. Thus $D(\lambda)$ is determined uniquely by the absolutely convergent series (14).

The function $D(x, y; \lambda)$ will be defined by the series

$$D(x, y; \lambda) = K(x, y) - \lambda \int \begin{vmatrix} K(x, y)\, dK(x, \xi) \\ K(\xi, y)\, dK(\xi, \xi) \end{vmatrix} + \frac{\lambda^2}{2!} \int\int \begin{vmatrix} K(x, y)\, dK(x, \xi)\, dK(x, \eta) \\ K(\xi, y)\, dK(\xi, \xi)\, dK(\xi, \eta) \\ K(\eta, y)\, dK(\eta, \xi)\, dK(\eta, \eta) \end{vmatrix} - \cdots$$

This involves integrals such as (15) and (16) and also integrals of the type

$$\int\int K(\eta, y)\, dK(\xi, \eta)\, dK(x, \xi) = \lim_{\|\Delta\xi_i\| \text{ etc.} \to 0} \sum_{i,j=1}^{m,n} K(\eta_j, y)\, \Delta K(\xi_i, \eta_j)\, \Delta K(x, \xi_i). \tag{21}$$

Since $K(x, a) = 0$, the right member of this equation is equivalent to

$$\lim \sum_{i,j,k=1}^{m,n,p} \Delta K(\eta_j, y_k)\, \Delta K(\xi_i, \eta_j)\, \Delta K(x, \xi_i),$$

where $a = y_0 < y_1 \cdots < y_p = y$. The proof that this is determined uniquely is essentially the same as that for the existence of the integral (16), and will not be repeated. The integral (21) is equal to each of the iterated integrals

$$\int \left[\int K(\eta, y)\, d\, K(\xi, \eta) \right] d\, K(x, \xi), \qquad \int K(\eta, y)\, d_\eta \int K(\xi, \eta)\, d\, K(x, \xi).$$

The series for $D(x, y; \lambda)$ can be proved to be uniformly convergent for any finite value of λ by an argument similar to that for the convergence of $D(\lambda)$. The expression $V_2 D(x, y; \lambda)$ cannot be greater than the least upper bound of a series whose general terms are in the form

$$\frac{\lambda^m}{m!} \sum_{i=1}^{m} \left| \sum_{j,\ldots,l=1}^{n,\ldots,q} \begin{vmatrix} \Delta K(x_i, y_i) & \Delta K(x_i, \xi_j) & \cdots & \Delta K(x_i, \zeta_l) \\ \Delta K(\xi_j, y_i) & \Delta K(\xi_j, \xi_j) & \cdots & \Delta K(\xi_j, \zeta_l) \\ \cdots & \cdots & \cdots & \cdots \\ \Delta K(\zeta_l, y_i) & \Delta K(\zeta_l, \xi_j) & \cdots & \Delta K(\zeta_l, \zeta_l) \end{vmatrix} \right|,$$

which cannot be greater than the expression

$$\frac{(m+1)^{\frac{m+1}{2}}\, \lambda^m}{m!} (V_2 K)^{m+1}.$$

As the series of which this is the general term is absolutely convergent for every finite value of λ, $V_2 D(x, y; \lambda)$ must be finite. This implies that $V_y D(x, y; \lambda)$ is bounded uniformly in x for every finite value of λ. The function $k(x, y; \lambda)$, defined by the equation

$$k(x, y; \lambda) = \frac{-D(x, y; \lambda)}{D(\lambda)},$$

will have the same properties for every value of λ for which $D(\lambda) \neq 0$.

If the expression

$$\varphi(x) = f(x) - \lambda \int f(y)\, d k(x, y; \lambda) \tag{22}$$

is substituted in equation (1), and the order of integration reversed in the iterated integral involved, the resulting equation is equivalent to

$$\int f(y)\, d_y \left[K(x,y) + k(x,y;\lambda) - \lambda \int k(\xi, y; \lambda)\, dK(x,\xi) \right] = 0.$$

The proof that

$$K(x,y) + k(x,y;\lambda) - \lambda \int k(\xi, y; \lambda)\, dK(x,\xi) = 0$$

is essentially the same as in the classical Fredholm theory, and will not be given here.* It also follows from equation (27) with $n = 1$. Thus the expression (22) is the required solution.

3. **The homogeneous equation.** The Fredholm minors for equation (1) will be defined in terms of the following;

$$\int \cdots \int K \begin{pmatrix} x_1 \cdots x_n\, ds_1 \cdots ds_p \\ y_1 \cdots y_n\, ds_1 \cdots ds_p \end{pmatrix}$$

(23)

$$= \int \cdots \int \begin{vmatrix} K(x_1, y_1) \cdots K(x_1, y_n)\, dK(x_1, s_1) \cdots dK(x_1, s_p) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ K(x_n, y_1) \cdots K(x_n, y_n)\, dK(x_n, s_1) \cdots dK(x_n, s_p) \\ K(s_1, y_1) \cdots K(s_1, y_n)\, dK(s_1, s_1) \cdots dK(s_1, s_p) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ K(s_p, y_1) \cdots K(s_p, y_n)\, dK(s_p, s_1) \cdots dK(s_p, s_p) \end{vmatrix}.$$

These multiple integrals can be proved to be approached uniformly in the same way as the integrals involved in $D(x, y; \lambda)$, and the iterated integrals involved are independent of the order of integration.

If the determinant in the right member of (23) is expanded with respect to the top row, the integral becomes equivalent to

$$\sum_{i=1}^{n} (-1)^{i-1} K(x_1, y_i) \int \cdots \int K \begin{pmatrix} x_2 \cdots x_n\, ds_1 \cdots ds_p \\ y_1 \cdots y_{i-1}\, y_{i+1} \cdots y_n\, ds_1 \cdots ds_p \end{pmatrix}$$

(24)

$$- p \int \left[\int \cdots \int K \begin{pmatrix} s x_2 \cdots x_n\, ds_1 \cdots ds_{p-1} \\ y_1 \cdots y_n\, ds_1 \cdots ds_{p-1} \end{pmatrix} \right] dK(x_1, y),$$

* Bôcher, loc. cit., p. 35 and p. 39.

and, if it is expanded with respect to the first column, the integral becomes equivalent to

$$\sum_{i=1}^{n}(-1)^{i-1}K(x_i, y_1)\int\cdots\int K\begin{pmatrix}x_1\cdots x_{i-1}\,x_{i+1}\cdots x_n\,ds_1\cdots ds_p\\ y_2\cdots\cdots\cdots\cdots y_n\,ds_1\cdots ds_p\end{pmatrix}$$

(25)

$$-p\int K(s, y_1)\,ds\int\cdots\int K\begin{pmatrix}x_1\cdots x_n\,ds_1\cdots ds_{p-1}\\ s\,y_2\cdots y_n\,ds_1\cdots ds_{p-1}\end{pmatrix}.^{*}$$

The Fredholm minor of the n-th order can now be defined by the equation

$$D\left(\begin{matrix}x_1\cdots x_n\\ y_1\cdots y_n\end{matrix}\middle|\,\lambda\right) = K\begin{pmatrix}x_1\cdots x_n\\ y_1\cdots y_n\end{pmatrix} - \lambda\int K\begin{pmatrix}x_1\cdots x_n\,ds_1\\ y_1\cdots y_n\,ds_1\end{pmatrix}$$

(26)

$$+\frac{\lambda^2}{2!}\int\int K\begin{pmatrix}x_1\cdots x_n\,ds_1\,ds_2\\ y_1\cdots y_n\,ds_1\,ds_2\end{pmatrix}-\cdots.$$

The proof that this is absolutely convergent and that its variation in any of the y's is bounded uniformly, is essentially the same as for $D(x, y; \lambda)$, which is the same as $D\left(\begin{matrix}x\\ y\end{matrix}\middle|\,\lambda\right)$. It also belongs to $\{f(x)\}$ when arbitrary values are assigned to all of the y's and to all but one of the x's, and the equation $K(x, a) = 0$ implies that it vanishes when any one of the y's is equal to a.

If the expressions (25) and (26) are substituted successively in equation (26), it gives the two equations

$$D\left(\begin{matrix}x_1\cdots x_n\\ y_1\cdots y_n\end{matrix}\middle|\,\lambda\right) = \sum_{i=1}^{n}(-1)^{i-1}\,D\left(\begin{matrix}x_2\cdots\cdots\cdots x_n\\ y_1\cdots y_{i-1}\,y_{i+1}\cdots y_n\end{matrix}\middle|\,\lambda\right)K(x_1, y_i)$$

(27)

$$+\lambda\int D\left(\begin{matrix}s\,x_2\cdots x_n\\ y_1\cdots y_n\end{matrix}\middle|\,\lambda\right)dK(x_1, s),$$

and

$$D\left(\begin{matrix}x_1\cdots x_n\\ y_1\cdots y_n\end{matrix}\middle|\,\lambda\right) = \sum_{i=1}^{n}(-1)^{i-1}\,K(x_i, y_1)\,D\left(\begin{matrix}x_1\cdots x_{i-1}\,x_{i+1}\cdots x_n\\ y_2\cdots\cdots\cdots\cdots y_n\end{matrix}\middle|\,\lambda\right)$$

(28)

$$+\lambda\int K(s, y_1)\,ds\,D\left(\begin{matrix}x_1\cdots x_n\\ s\,y_2\cdots y_n\end{matrix}\middle|\,\lambda\right).$$

* Compare with Goursat, Cours d'Analyse Mathématique, vol. 3, pp. 375–7.

It follows from equation (27) that if all of the Fredholm minors of degree less than n vanish identically for a given value of λ, while

$$D\begin{pmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{pmatrix} \neq 0$$

for a given set of values of its arguments, the function

$$\varphi_1(x) = D\begin{pmatrix} x\, x_2 \cdots x_n \\ y_1 \cdots\cdots y_n \end{pmatrix}$$

is a solution of the homogeneous equation

$$\varphi(x) = \lambda \int \varphi(y)\, d\, K(x, y). \tag{29}$$

Similarly the functions

$$\begin{aligned} \varphi_i(x) &= D\begin{pmatrix} x_1 \cdots x_{i-1}\, x\, x_{i+1} \cdots x_n \\ y_1 \cdots\cdots\cdots\cdots y_n \end{pmatrix} \\ &= (-1)^{i-1} D\begin{pmatrix} x\, x_1 \cdots x_{i-1}\, x_{i+1} \cdots x_n \\ y_1 \cdots\cdots\cdots\cdots y_n \end{pmatrix}, \quad (i = 1, 2, \ldots, n), \end{aligned} \tag{30}$$

all satisfy the same equation.

The proof that these solutions are linearly independent is essentially the same as in the classical theory.*

If $\varphi(x)$ is an arbitrary solution of equation (29) and $H(x, y)$ is defined by the equation

$$H(x, y) = \frac{D\begin{pmatrix} x\, x_1 \cdots x_n \\ y\, y_1 \cdots y_n \end{pmatrix}}{D\begin{pmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{pmatrix}}, \tag{31}$$

the equations

$$\begin{aligned} \lambda \int \varphi(y)\, d\, H(x, y) &= \lambda^2 \int \left[\int \varphi(s)\, d\, K(y, s) \right] d\, H(x, y) \\ &= \lambda^2 \int \varphi(y)\, d_s \int K(s, y)\, d\, H(x, s) \end{aligned}$$

* Goursat, loc. cit., pp. 378–9.

must be satisfied. If this is added to equation (29), and one term transposed, it becomes

$$(32) \quad \varphi(x) = \lambda \int \varphi(y)\, d_y \Big[K(x,y) - H(x,y) + \lambda \int K(s,y)\, d\, H(x,s) \Big].$$

Equation (28) applied to the numerator of equation (31), makes $H(x,y)$ satisfy the equation

$$H(x,y) = K(x,y) - \sum_{i=1}^{n} \frac{\varphi_i(x)\, K(x_i,y)}{D\begin{pmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{pmatrix} \lambda} + \lambda \int K(s,y)\, d\, H(x,s).$$

When this is substituted for $H(x,y)$ in equation (32) it makes $\varphi(x)$ equal to a linear combination of the functions (30). Thus every solution of equation (29) is a linear combination of these particular solutions.

4. **Independence of the Riesz Theory.** The equation

$$f(x) = \varphi(x) - \lambda\, T(\varphi)$$

has been discussed by Riesz, where $T(f)$ is a completely continuous transformation. This is equivalent to equation (1) if $V_y\, K(x,y)$ is bounded uniformly, and if, whenever a sequence $x_1, x_2, \ldots$ is chosen in such a way that $K(x_n, y)$ approaches a unique function of y as n becomes infinite. the equation

$$(33) \qquad \lim_{n \to \infty} V_y [K(x_n,y) - \lim_{n \to \infty} K(x_n,y)] = 0^*$$

is satisfied. In order to show that these conditions are not equivalent to those assumed in the last two sections, an example will be given where $K(x,y)$ satisfies the conditions stated in § 2, but does not satisfy equation (33), and a second example where $V_y\, K(x,y)$ is bounded uniformly and equation (33) is satisfied, but where $\int d\, K(x,x)$ is infinite.

If $K(x,y)$ is defined as

$$K(x,y) = x \sin \frac{y}{x}, \qquad (0 < x \leqq \pi;\ 0 \leq y \leq \pi),$$
$$K(0,y) = 0,$$

* Fischer, loc. cit.

the inequality

$$\left| \frac{\partial K(x, y)}{\partial y} \right| \leqq 1,$$

will be satisfied for all values of x and y. Consequently $V_2 K(x, y)$ cannot be greater than π, and equation (18) is satisfied identically in x_a. The only x_a needed, however, is $x_1 = 0$, and since

$$\left| \frac{\partial^2 K(x, y)}{\partial x \partial y} \right| \leq \frac{\pi}{x^2},$$

the inequality

$$V_\delta K(x, y)\Big|_{R'} < \frac{\delta \pi^2}{\epsilon'^2},$$

must be satisfied, where R' is the region $(\epsilon' < x \leq b;\ a \leqq y \leq b)$, and inequality (17) will be satisfied if δ is small enough. Thus the hypothesis in § 2 is satisfied. But the sequence $K(l/n, y)$ approaches zero as n becomes infinite, while

$$\lim_{n \to \infty} V_y [K(1/n, y) - \lim_{n \to \infty} K(1/n, y)] = 2,$$

and the Riesz theory does not apply.

On the other hand, if $K(x, y)$ is defined by the equation

$$K(x, y) = (y - x)^{\frac{1}{3}}, \qquad (0 \leqq x \leqq 1;\ 0 \leqq y \leqq 1),$$

$V_y K(x, y)$ must be less than 2. Also if $K(x_n, y)$ converges as n becomes infinite, x_n must approach a unique limit, which will be called c, and the limit of $K(x_n, y)$ will be $K(c, y)$. It follows from this definition of $K(x, y)$ that

$$\frac{\partial}{\partial y} [K(x_n, y) - K(c, y)] = \frac{(y - c)^{\frac{2}{3}} - (y - x_n)^{\frac{2}{3}}}{3 (y - x_n)^{\frac{2}{3}} (y - c)^{\frac{2}{3}}}.$$

If ϵ is chosen arbitrarily small, this partial derivative approaches zero uniformly in each of the intervals $(0 \leqq y \leqq c - \epsilon)$ and $(c - \epsilon \leqq y \leqq 1)$, as n becomes infinite. Consequently for n large enough, the inequality

$$V_y [K(x_n, y) - K(c, y)] < \epsilon$$

must be satisfied. If this is added to equation (29), and one term transposed, it becomes

$$(32) \quad \varphi(x) = \lambda \int \varphi(y) d_y \Big[K(x,y) - H(x,y) + \lambda \int K(s,y)\, d\, H(x,s) \Big].$$

Equation (28) applied to the numerator of equation (31), makes $H(x,y)$ satisfy the equation

$$H(x,y) = K(x,y) - \sum_{i=1}^{n} \frac{\varphi_i(x)\, K(x_i, y)}{D\begin{pmatrix} x_1 \cdots x_n \\ y_1 \cdots y_n \end{pmatrix} \lambda} + \lambda \int K(s,y)\, d\, H(x,s).$$

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$$V_\delta K(x, y)\Big|_{R'} < \frac{\delta \pi^2}{\epsilon'^2},$$

must be satisfied, where R' is the region $(\epsilon' < x \leqq b;\ a \leqq y \leqq b)$, and inequality (17) will be satisfied if δ is small enough. Thus the hypothesis in § 2 is satisfied. But the sequence $K(l/n, y)$ approaches zero as n becomes infinite, while

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On the other hand, if $K(x, y)$ is defined by the equation

$$K(x, y) = (y - x)^{\frac{1}{3}}, \qquad (0 \leqq x \leqq 1;\ 0 \leqq y \leqq 1),$$

$V_y K(x, y)$ must be less than 2. Also if $K(x_n, y)$ converges as n becomes infinite, x_n must approach a unique limit, which will be called c, and the limit of $K(x_n, y)$ will be $K(c, y)$. It follows from this definition of $K(x, y)$ that

$$\frac{\partial}{\partial y} [K(x_n, y) - K(c, y)] = \frac{(y - c)^{\frac{2}{3}} - (y - x_n)^{\frac{2}{3}}}{3 (y - x_n)^{\frac{2}{3}} (y - c)^{\frac{2}{3}}}.$$

If ϵ is chosen arbitrarily small, this partial derivative approaches zero uniformly in each of the intervals $(0 \leq y \leq c - \epsilon)$ and $(c - \epsilon \leqq y \leqq 1)$, as n becomes infinite. Consequently for n large enough, the inequality

$$V_y [K(x_n, y) - K(c, y)] < \epsilon$$

must be satisfied in each of the above intervals. But the inequality

$$V_y[K(x_n, y) - K(c, y)] < 4\varepsilon^{\frac{1}{3}}, \qquad (c - \varepsilon \leq y \leq c + \varepsilon),$$

must be satisfied for every value of n. Then, if n is large enough,

$$V_y[K(x_n, y) - K(c, y)] < 2\varepsilon + 4\varepsilon^{\frac{1}{3}}, \qquad (0 \leq y \leq 1).$$

Since this expression approaches zero with ε, equation (33) must be satisfied, and the Riesz theory applies to this kernel. But if Δx_i is taken equal to $1/n$, the equation

$$\sum_{i=1}^{n} \Delta K(x_i, x_i) = n^{\frac{2}{3}}$$

will be satisfied. This makes the integral (15) infinite, and the Fredholm determinant, as defined by equation (14), does not exist.

TRINITY COLLEGE,
HARTFORD, CONN.

RECTILINEAR CONGRUENCES REFERRED TO SPECIAL SURFACES.

By Malcolm Cecil Foster.

Introduction. The aim of this paper is to study the properties of rectilinear congruences by the method of the moving trihedral. A one-to-one correspondence is established between the lines of the congruence and the points of a surface by defining each line with reference to the trihedral at the corresponding point.

When each line of the congruence is given its most general orientation relative to the trihedral at the associated point, and the surface of reference and the parametric system upon it are any whatever, the derivation of the fundamental equations and conditions for the study of congruences is very laborious and the results very cumbersome. We may without loss of generality choose the unit sphere for the surface of reference, and associate each line of the congruence with that point on the sphere through which the positive normal has the same direction. With this very special orientation, and when the parametric system on the sphere is any isothermal system, the fundamental equations and conditions are greatly simplified.

Many of the results contained in this paper are well known; those results which are believed to be new are stated explicitly as theorems.

This paper is an extract from the author's thesis presented to the Faculty of the Graduate School of Yale University in candidacy for the degree of Doctor of Philosophy.

A. Rectilinear congruences referred to the unit sphere.

1. We refer the sphere to any isothermal system, and take the x-axis of the trihedral tangent to the curve $v = \text{const.}$ The equations of any line l referred to its trihedral are

$$x = a, \quad y = b, \quad z = t, \tag{1}$$

where the length t of l is measured from the point $(a, b, 0)$, and a and b are functions of u and v. We write the linear element of the sphere in the form

$$ds^2 = e^{2\lambda}\,(du^2 + dv^2). \tag{2}$$

For the sphere, $F = D' = 0$.

Hence

$$\xi_1 = \eta = p = q_1 = 0.^* \tag{3}$$

Since $E = \mathfrak{E}$, $G = \mathfrak{G}$, and $\varrho_1 = \varrho_2 = -1$,† we have $D = -E$, $D'' = -G = -E$. Also

$$q = -\frac{D}{\sqrt{E}} = \sqrt{E}, \quad p_1 = \frac{D''}{\sqrt{G}} = -\sqrt{E},$$

$$r = -\frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial v}, \qquad r_1 = \frac{1}{\sqrt{E}}\frac{\partial\sqrt{E}}{\partial u}. \tag{4}$$

From (2) and (4)

$$\begin{aligned} p_1 &= -e^{\lambda}, & r &= -\lambda_v, & \xi &= e^{\lambda}, \\ q &= e^{\lambda}, & r_1 &= \lambda_u, & \eta_1 &= e^{\lambda}. \end{aligned} \tag{5}$$

When these values are put in the six fundamental relations corresponding to the Gauss and Codazzi equations,‡ all except the equation

$$\lambda_{uu} + \lambda_{vv} = -e^{2\lambda}, \tag{6}$$

are identically satisfied. The general solution of (6) is

$$\lambda = \log \frac{(\psi_u^2 + \psi_v^2)^{\frac{1}{2}}}{\cosh \psi}, \tag{7}$$

where ψ is of the form

$$\psi = \Phi_1(u + iv) + \Phi_2(u - iv), \tag{8}$$

and Φ_1, and Φ_2 conjugate functions for ψ real. Every real solution of (6) defines an isothermal system on the sphere.

* The notation used here is the same as in Eisenhart's "Differential Geometry of Curves and Surfaces," pp. 166-176.

† The functions $\mathfrak{E}$, $\mathfrak{F}$, $\mathfrak{G}$, are the fundamental coefficients for the spherical representation of the congruence. The positive direction on the normal is chosen outward.

‡ Eisenhart, p. 168 and p. 170.

2. **Condition for a normal congruence.** As the vertex of the trihedral is displaced on the surface along a curve defined by a value of dv/du, a point $P(x, y, z)$, receives in the direction of the axes of the trihedral the absolute displacements,*

$$
\begin{aligned}
\delta x &= dx + \xi\, du + \xi_1\, dv + (q\, du + q_1 dv)z - (r\, du + r_1 dv)y, \\
\delta y &= dy + \eta\, du + \eta_1\, dv + (r\, du + r_1 dv)x - (p\, du + p_1 dv)z, \\
\delta z &= dz \qquad\qquad\qquad + (p du + p_1 dv)y - (q\, du + q_1 dv)x.
\end{aligned} \tag{9}
$$

In order that a congruence be normal, the displacement of $P(x, y, z)$ must be such that $\delta z = 0$ for all values of dv/du. Hence from (1) and (9)

$$t_u\, du + t_v\, dv + b p_1\, dv - a q\, du = 0.$$

Then

$$t_u = a q = a e^{\lambda}, \qquad t_v = -b p_1 = b e^{\lambda}.$$

The condition of integrability is

$$\frac{\partial}{\partial v}(a e^{\lambda}) = \frac{\partial}{\partial u}(b e^{\lambda}),$$

which may be replaced by

$$a = e^{-\lambda} P_u, \qquad b = e^{-\lambda} P_v, \tag{10}$$

where P is an arbitrary function of u and v. Hence (10) is the condition that a congruence be normal. Conversely, for any arbitrary function P, the values of a and b as given by (10) define a normal congruence with a single infinity of normal surfaces.

When these values of a and b are substituted for x and y respectively in the equation $\delta z = 0$, we get

$$dz - P_u\, du - P_v\, dv = 0.$$

Hence

$$z = P; \tag{11}$$

that is, P is the distance to a normal surface.

* Eisenhart, p. 170.

3. **Equation for the developables. Equation for the distances to the focal points.** There must be a point (x, y, z), such that when the displacement is in the developable $\delta x = 0$, and $\delta y = 0$. These equations become

$$a_u\,du + a_v\,dv + \xi\,du + tq\,du - br\,du - br_1\,dv = 0,$$

$$b_u\,du + b_v\,dv + \xi\,dv + ar\,du + ar_1\,dv - tp_1\,dv = 0.$$

The elimination of t from these equations gives the following equation for the developables:

$$(12) \qquad -(b_u + ar)\,du^2 + (a_u - b_v - br - ar_1)\,du\,dv + (a_v - br_1)\,dv^2 = 0.$$

The elimination of the ratio dv/du gives

$$t^2(p_1 q) + t[p_1(a_u + \xi - br) - q(b_v + \xi + ar_1)]$$
$$(13) \qquad -(a_u + \xi - br)(b_v + \xi + ar_1) + (a_v - br_1)(b_u + ar) = 0$$

for the equation for the distances to the focal points.

4. **Equation for the principal surfaces. Equation for the distances to the limit points.** Let us consider the variations which the direction-cosines (0, 0, 1) experience as the vertex M of the trihedral is displaced along a curve C to M'. From the center of the sphere we draw lines parallel to the axes of the trihedral, forming a second trihedral T_0 which moves so that its axes are in every position parallel to the axes of the trihedral on the sphere. From the vertex of T_0 we draw a line parallel to the corresponding line of the congruence. Relative to T_0, the coördinates of the point in which this ine pierces the sphere are (0, 0, 1). The displacements of this point in the directions of the axes of T_0 will be the variations experienced by the direction-cosines of l as M is displaced to M'. Let l' be the line of the congruence corresponding to M', and PP' be the common perpendicular to l and l'. The displacement of the limiting position of the foot of this common perpendicular on l must be orthogonal to both l and l'. Hence from (9), since the translations ξ and η_1 are zero for the motion of T_0,

$$q\,du\,(a_u\,du + a_v\,dv + \xi\,du + tq\,du - br\,du - br_1\,dv)$$
$$-p_1\,dv\,(b_u\,du + b_v\,dv + \xi\,dv + ar\,du + ar_1\,dv - tp_1\,dv) = 0.$$

Setting $dv/du = k$, and solving for t, we get

$$t = -\frac{a_u + \xi - br + k(a_v + b_u + ar - br_1) + k^2(b_v + \xi + ar_1)}{q + qk^2}. \tag{14}$$

For the principal surface we set $dt/dk = 0$. This gives

$$(a_v + b_u + ar - br_1)(du^2 - dv^2) + 2(b_v - a_u + ar_1 + br)\,du\,dv = 0. \tag{15}$$

Hence (15) is the equation of the principal surfaces. Since the coefficients of du^2 and dv^2 are equal and opposite in sign, we have the well known theorem:

The principal surfaces are represented on the sphere by an orthogonal system.

From (14) and (15)

$$t = \frac{a_u + \xi - br + \frac{k}{2}(a_v + b_u + ar - br_1)}{-q}$$

$$= \frac{\frac{1}{2}(a_v + b_u + ar - br_1) + k(b_v + \xi + ar_1)}{-qk}.$$

Hence the following equations hold between the distances t_1 and t_2 to the limit points, and the corresponding values of k:

$$a_u + \xi - br + \frac{k}{2}(a_v + b_u + ar - br_1) + qt = 0,$$

$$\frac{1}{2}(a_v + b_u + ar - br_1) + k(b_v + \xi + ar_1) + qkt = 0.$$

When k is eliminated from these equations we get the following equation for the distances to the limit points:

$$4q^2t^2 + 4q(a_u + b_v + ar_1 - br + 2\xi)t$$
$$+ 4(a_u + \xi - br)(b_v + \xi + ar_1) - (a_v + b_u + ar - br_1)^2 = 0. \tag{16}$$

Hence

$$t_1 + t_2 = \frac{a_u + b_v + ar_1 - br + 2\xi}{p_1}. \tag{17}$$

Since from (13), the sum of the distances to the focal points is identical with (17), we have the known theorem:

On any line of a congruence, the two segments bounded respectively by the focal points and the limit points are so placed as to have the same midle point.

If the congruence be normal, equations (12) and (15) become identical on using (10). Hence we have the well known theorem:

The principal surfaces are developable when the congruence is normal.

For a normal congruence equations (13) and (16) are identical; hence the focal points and the limit points coincide when the congruence is normal.

From (12), the condition that the curves defining the developables form an orthogonal system is

$$a_v - b_u - ar - br_1 = 0.$$

Since this is identical with the condition that the congruence be normal, we have the known theorem:

A necessary and sufficient condition that the focal planes be perpendicular, is that the congruence be normal.

5. **Condition for an isotropic congruence.** The necessary and sufficient condition that a congruence be isotropic is that its developables be represented on the sphere by the minimal lines. Hence from (12), we must have

$$-b_u - ar = a_v - br_1, \qquad a_u - b_v - br - ar_1 = 0.$$

If we substitute the values of r and r_1 in these equations, and multiply by $e^{-\lambda}$, we get

$$(18) \qquad \frac{\partial}{\partial v}(ae^{-\lambda}) = -\frac{\partial}{\partial u}(be^{-\lambda}), \qquad \frac{\partial}{\partial u}(ae^{-\lambda}) = \frac{\partial}{\partial v}(be^{-\lambda}).$$

From the first of equations (18) it follows that we may define a function Q by the equations

$$ae^{-\lambda} = Q_u, \qquad be^{-\lambda} = Q_v;$$

and from the second of equations (18), it follows that Q must satisfy the equation

$$Q_{uu} + Q_{vv} = 0.$$

Hence the necessary and sufficient condition that a congruence be isotropic is that

$$a = e^{\lambda} Q_u, \qquad b = -e^{\lambda} Q_v,$$

$$Q_{uu} + Q_{vv} = 0. \tag{19}$$

6. **Condition that the lines of curvature on the normal surfaces be represented by an isothermal system on the sphere.** Using the values of a and b from (10) and (12), the equation of the developables becomes

$$(P_{uv} - \lambda_u P_v - \lambda_v P_u)(du^2 - dv^2) + (P_{vv} - P_{uu} + 2\lambda_u P_u - 2\lambda_v P_v)\, du\, dv = 0.$$

Hence the condition that the lines of curvature on the normal surfaces be represented on the sphere by the chosen isothermal system is that

$$P_{uv} - \lambda_u P_v - \lambda_v P_u = 0.^{*} \tag{20}$$

7. **Congruences whose middle envelope is a point.** Without loss of generality we may take the origin as the middle envelope. The condition that the middle envelope be the origin is therefore

$$\frac{t_1 + t_2}{2} = -1,$$

where t_1 and t_2 are the roots of (13). From (13) we have

$$\frac{a_u + b_v + a r_1 - b r + 2\xi}{2 p_1} = -1,$$

which reduces to

$$a_u + b_v + a\lambda_u + b\lambda_v = 0. \tag{21}$$

Multiplying by e^{λ}, we get

$$\frac{\partial}{\partial u}(a e^{\lambda}) = -\frac{\partial}{\partial v}(b e^{\lambda}) = R_{uv},$$

* We note from § 2 that in the notation usually employed in tangential coördinates, the distance W from the origin or center of the sphere to the tangent plane of the normal surfaces is $P+1$. The condition (20) is identical with the well known condition satisfied by W in order that $D' = 0$ for the normal surface. Cf. Eisenhart, p. 164, equations (34).

where R is any function of u and v. Hence for a congruence whose middle envelope is the origin

$$a = e^{-\lambda} R_v, \qquad b = -e^{-\lambda} R_u. \tag{22}$$

We have seen that for a normal congruence

$$a = e^{-\lambda} P_u, \qquad b = e^{-\lambda} P_v.$$

If each line of this normal congruence be rotated in a positive* direction through an angle $\pi/2$ about the corresponding normal, the rotated line will pierce the tangent plane at the point

$$a = -e^{-\lambda} P_v, \qquad b = e^{-\lambda} P_u.$$

Since (21) is satisfied identically for these values of a and b we have:

THEOREM 1. *If each line of a normal congruence be rotated through an angle $\pi/2$ about the corresponding normal to the sphere, the middle envelope of the resulting congruence is the origin. Conversely, all congruences for which the middle envelope is the origin are determined in this way.*

Suppose each line of the congruence defined by (22) be rotated through an angle $\pi/2$ in the positive direction about the corresponding normal. The line in its new position will pierce the tangent plane at the point $(e^{-\lambda} R_u, e^{-\lambda} R_v)$. When these values of a and b are put in (21), we get

$$e^{-\lambda}(R_{uu} + R_{vv}) = 0.$$

Hence we have:

THEOREM 2. *If the function R is a solution of Laplace's equation, and if each line of the congruence defined by (22) be rotated through an angle $\pi/2$ about the corresponding normal to the sphere, the resulting congruence is normal and such that its middle envelope is likewise the origin.*

8. Let us consider the normal congruence defined by

$$a = e^{-\lambda} P_u, \qquad b = e^{-\lambda} P_v,$$

and the congruence formed by rotating each line through an angle $\pi/2$ in the positive direction about the corresponding normal. We shall call the new congruence, the rotated congruence; it is defined by

$$a_1 = -e^{-\lambda} P_v, \qquad b_1 = e^{-\lambda} P_u.$$

* If each line be rotated negatively, the resulting congruence will be determined by $a = e^{-\lambda} P_v$, $b = -e^{-\lambda} P_u$. Hence from (22), the direction of rotation is immaterial.

From (12), the equation of the developables of the normal congruence is

$$(23)\quad (P_{uv}-\lambda_u P_v-\lambda_v P_u)(dv^2-du^2)+(P_{uu}-P_{vv}-2\lambda_u P_u+2\lambda_v P_v)\,du\,dv=0;$$

and from (15), the equation of the principal surfaces of the rotated congruence is

$$(24)\quad (P_{uu}-P_{vv}-2\lambda_u P_u+2\lambda_v P_v)(du^2-dv^2)+4(P_{uv}-\lambda_u P_v-\lambda_v P_u)\,du\,dv=0.$$

Comparing (23) and (24), we may write these equations in the following abbreviated form:

$$-m\,du^2+k\,du\,dv+m\,dv^2=0,$$

$$k\,du^2+4\,m\,du\,dv-k\,dv^2=0.$$

The curves in each family form an orthogonal system. We shall now determine the angles between the curves defined by (23) and (24). If θ_1 and θ_2 denote the angles between the positive directions of the curve $v=$ const. and the curves defined by (23) and (24) respectively, we have

$$(25)\quad \tan\theta_1=\frac{H}{E}\frac{dv}{du}=\frac{dv}{du}=\frac{-k\pm\sqrt{k^2+4m^2}}{2m},$$

$$\tan\theta_2=\frac{-2m\pm\sqrt{k^2+4m^2}}{-k}.$$

Let θ denote the angle determined by the upper signs in (25). Then

$$\tan\theta=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2},$$

which from (25) reduces to

$$\tan\theta=1;$$

hence $\theta=\pi/4$, and we have:

THEOREM 3. *The curves defining the developables of a normal congruence and the principal surfaces of the rotated congruence form orthogonal systems and bisect each other.*

We also note that if the developables of the normal congruence are defined by an isothermal system, then so also are the principal surfaces of the rotated congruence.

9. Consider again a normal congruence and the rotated congruence. We ask: When is the rotated congruence an isotropic congruence?

Equation (24), the equation of the principal surfaces for the rotated congruence must be satisfied identically.* But if (24) is satisfied identically, so also is (23), the equation of the developables of the normal congruence. That is, the equation of the lines of curvature on the normal surfaces is satisfied identically; hence the normal congruence consists of the normals to a sphere or a plane. Hence we have the theorem:

THEOREM 4. *The necessary and sufficient condition that the congruence formed by rotating a normal congruence be isotropic is that the normal congruence consist of the lines through a point, or the normals to a plane.*

10. For an isotropic congruence referred to the sphere

$$a = e^{\lambda} Q_u, \qquad b = -e^{\lambda} Q_v,$$

$$Q_{uu} + Q_{vv} = 0. \tag{26}$$

If each line be rotated through an angle $\pi/2$ in the positive direction about the corresponding normal, the resulting congruence will be defined by

$$a_1 = e^{\lambda} Q_v, \qquad b_1 = e^{\lambda} Q_u.$$

Since Q is a solution of Laplace's equation, we set

$$K_u = Q_v, \qquad K_v = -Q_u,$$

where K and Q are the real and imaginary parts respectively of an analytic function of $u + iv$. Hence

$$a_1 = e^{\lambda} K_u, \qquad b_1 = -e^{\lambda} K_v.$$

Since K also is a solution of Laplace's equation, we have by comparing these values of a_1 and b_1 with (26).

THEOREM 5. *If each line of an isotropic congruence referred to the sphere be rotated through an angle $\pi/2$ about the corresponding normal,† the resulting congruence is also isotropic.*‡

* Eisenhart, p. 413.

† The direction of rotation is immaterial; for if each line is rotated negatively, the effect is to change the sign of Q.

‡ Eisenhart, p. 423, Ex. 9.

B. Rectilinear congruences referred to minimal surfaces whose lines of curvature are parametric.

11. We now refer any congruence to a minimal surface by establishing a one-to-one correspondence between the lines of the congruence and the points of the surface at which the z-axis of the trihedral is parallel to the line, and such that their positive directions are the same. The lines of curvature form an isothermal system. Without loss of generality we choose*

$$D = -D'' = 1.$$

Then $E = G = \varrho$, and $\mathfrak{E} = \mathfrak{G} = 1/\varrho$, where $\varrho = |\varrho_1| = |\varrho_2|$; also $F = \mathfrak{F} = D' = 0$. The linear elements of the surface and the spherical representation are respectively

$$ds^2 = \varrho(du^2 + dv^2) \quad \text{and} \quad d\sigma^2 = \frac{1}{\varrho}(du^2 + dv^2).$$

We choose the x-axis of the trihedral tangent to the curve $v = \text{const.}$; hence

$$\xi_1 = \eta = p = q_1 = 0.$$

We set $\frac{1}{2}\log\varrho = -\lambda$; then

$$ds^2 = e^{-2\lambda}(du^2 + dv^2), \qquad d\sigma^2 = e^{2\lambda}(du^2 + dv^2),$$

$$\xi = \eta_1 = e^{-\lambda}, \qquad p_1 = q = -e^{\lambda}, \tag{27}$$

$$r = \lambda_v, \qquad r_1 = -\lambda_u, \qquad \mathfrak{E} = \mathfrak{G} = e^{2\lambda}.$$

When the values of $\xi, \eta_1, p_1, q, r, r_1$, from (27) are substituted in the six fundamental equations which are the equivalents of the Gauss and Codazzi equations, all except the equation

$$\lambda_{uu} + \lambda_{vv} = -e^{2\lambda}, \tag{28}$$

are identically satisfied. This equation is the same as (6). Hence any solution (7) of (28) determines a particular isothermal system on the sphere and a particular minimal surface.

12. Without repeating the reasoning as outlined in section A, we give here only the results.

* Eisenhart, pp. 253–254.

The necessary and sufficient condition for a normal congruence is

$$a = e^{-\lambda} \bar{P}_u, \qquad b = -e^{-\lambda} \bar{P}_v, \tag{29}$$

where P is any function of u and v. The distance to a normal surface is equal to $-\bar{P}$.

The equation for the developables is

$$(b_u + ar)\,du^2 + (a_u + b_v + ar_1 - br + 2\xi)\,du\,dv + (a_v - br_1)\,dv^2 = 0; \tag{30}$$

the equation for the distances to the focal points is

$$\begin{aligned} &q^2 t^2 + q(a_u - b_v - ar_1 - br)\,t + (a_v - br_1)(b_u + ar) \\ &\qquad - (a_u + \xi - br)(b_v + \xi + ar_1) = 0. \end{aligned} \tag{31}$$

The principal surfaces are defined by the equation

$$(a_v - b_u - ar - br_1)(du^2 - dv^2) - 2(a_u + b_v + ar_1 - br + 2\xi)\,du\,dv = 0; \tag{32}$$

the distances to the limit points are given by the equation

$$\begin{aligned} &4q^2 t^2 + 4q(a_u - b_v - ar_1 - br)\,t \\ &\qquad - \{4(a_u + \xi - br)(b_v + \xi + ar_1) + (a_v - b_u - ar - br_1)^2\} = 0. \end{aligned} \tag{33}$$

For an isotropic congruence,

$$\begin{aligned} &a = e^{\lambda} Q_u, \quad b = e^{\lambda} \bar{Q}_v, \\ &Q_{uu} + \bar{Q}_{vv} + 2e^{-2\lambda} = 0. \end{aligned} \tag{34}$$

12. If the congruence be normal the equation of the developables is

$$\begin{aligned} &(P_{uv} - \lambda_u \bar{P}_v - \lambda_v \bar{P}_u)(du^2 - dv^2) \\ &\qquad - (P_{uu} - P_{vv} - 2\lambda_u P_u + 2\lambda_v \bar{P}_v + 2)\,du\,dv = 0. \end{aligned} \tag{35}$$

Hence the condition that the lines of curvature on the normal surface be represented on the minimal surface by the lines of curvature, is

$$\bar{P}_{uv} - \lambda_u \bar{P}_v - \lambda_v \bar{P}_u = 0; \tag{36}$$

this condition is identical with (20).

Since at any point on a minimal surface, the asymptotic lines bisect the angles between the lines of curvature, we have from (35) that the necessary and sufficient condition that the lines of curvature on the normal surfaces be represented on the minimal surface by the asymptotic lines, is

$$\bar{P}_{uu} - \bar{P}_{vv} - 2\lambda_u \bar{P}_u + 2\lambda_v \bar{P}_v + 2 = 0. \tag{37}$$

The coefficient of t in (31) reduces to

$$-(\bar{P}_{uu} + \bar{P}_{vv});$$

hence the necessary and sufficient condition that the given minimal surface be the mean evolute is that $\bar{P}$ be a solution of Laplace's equation.

C. Rectilinear congruences referred to minimal surfaces whose asymptotic lines are parametric.

14. We again associate each line of the congruence with that point on a minimal surface at which the normal is parallel to the line, and choose the x-axis of the trihedral tangent to the curve $v = \text{const.}$ We refer the surface to its asymptotic lines; hence $D = D'' = 0$, and without loss of generality we take $D' = 1$. This minimal surface is the adjoint of the minimal surface just considered whose lines of curvature were parametric.*

The linear elements of the minimal surface and the spherical representation are respectively

$$ds^2 = \varrho\,(du^2 + dv^2), \qquad dv^2 = \frac{1}{\varrho}(du^2 + dv^2),$$

where $\varrho = |\varrho_1| = |\varrho_2|$. Then

$$\xi_1 = \eta = p_1 = q = 0.$$

We set $\frac{1}{2}\log\varrho = -\lambda$; hence

$$ds^2 = e^{-2\lambda}(du^2 + dv^2), \qquad dv^2 = e^{2\lambda}(du^2 + dv^2),$$

$$\xi = \eta_1 = e^{-\lambda}, \qquad p = e^{\lambda}, \qquad q_1 = -e^{\lambda}, \tag{38}$$

$$r = \lambda_v, \qquad r_1 = -\lambda_u, \qquad \mathfrak{E} = \mathfrak{G} = e^{2\lambda}.$$

* Eisenhart, pp. 253–254.

When these values of ξ, η_1, p, q_1, r, r_1, are put in the six fundamental equations which correspond to the Gauss and Codazzi equations, all except the equation

$$\lambda_{uu} + \lambda_{vv} = -e^{2\lambda},$$

are identically satisfied. This is the same equation in λ as the equation which appeared in the discussions of the two previous sections. Hence any solution of this equation determines a particular isothermal system on the sphere and two particular minimal surfaces which are the adjoints of one another. The solution

$$\lambda = \log \operatorname{sech} u$$

defines the meridians and parallels as the isothermal system on the sphere; it is readily shown that the two adjoint minimal surfaces determined by this value of λ are the catenoid and the right helicoid.*

15. Without repeating the methods of derivation, we collect here the fundamental equations for the developables and the principal surfaces, and the conditions upon a and b for special congruences.

The necessary and sufficient condition for a normal congruence is

$$a = e^{-\lambda} \bar{P}'_v, \qquad b = e^{-\lambda} \bar{P}'_u, \tag{39}$$

where $\bar{P}'$ is any function of u and v. The distance to a normal surface is

$$z = -\bar{P}'. \tag{40}$$

The equations for the developables and for the distances to the focal points are respectively

$$(a_u + \xi - br)\,du^2 + (a_v - b_u - ar - br_1)\,du\,dv - (b_v + \xi + ar_1)\,dv^2 = 0, \tag{41}$$

and

$$p^2 t^2 - p(a_v + b_u + ar - br_1)\,t + (a_v - br_1)(b_u + ar) - (a_u + \xi - br)(b_v + \xi + ar_1) = 0. \tag{42}$$

The equation for the principal surfaces is

$$(a_u + b_v + ar_1 - br + 2\xi)(du^2 - dv^2) + 2(a_v - b_u - ar - br_1)\,du\,dv = 0. \tag{43}$$

* Eisenhart, p. 267.

and the equation for the distances to the limit points is

$$4p^2t^2 - 4p(a_v + b_u + ar - br_1)t$$
$$+ 4(a_v - br_1)(b_u + ar) - (a_u + b_v + ar_1 - br + 2\xi)^2 = 0. \tag{44}$$

For an isotropic congruence,

$$a = e^{\lambda}\bar{Q}'_u, \qquad b = e^{\lambda}\bar{Q}'_v,$$
$$Q'_{uu} + \bar{Q}'_{vv} + 2e^{-2\lambda} = 0. \tag{45}$$

16. **Summary of conditions upon a and b.** We have seen that the conditions that a congruence be of a particular type are conditions upon the functions a and b. We collect here these results with the expressions for the fundamental quantities for the three surfaces considered.

	Sphere (S) Isothermal System	Minimal Surface (S_1) Lines of Curvature Parametric	Minimal Surface (S_2) Asymptotic Lines Parametric
Normal Congruence	$a = e^{-\lambda}P_u$ $b = e^{-\lambda}P_v$	$a = e^{-\lambda}\bar{P}_u$ $b = -e^{-\lambda}\bar{P}_v$	$a = e^{-\lambda}\bar{P}'_v$ $b = e^{-\lambda}\bar{P}'_u$
Distance to Normal Surface	P	$-\bar{P}$	$-\bar{P}'$
Isotropic Congruence	$a = e^{-\lambda}Q_u$, $b = -e^{-\lambda}Q_v$ $Q_{uu} + Q_{vv} = 0$	$a = e^{\lambda}\bar{Q}_u$, $b = e^{\lambda}\bar{Q}_v$ $\bar{Q}_{uu} + \bar{Q}_{vv} + 2e^{-2\lambda} = 0$	$a = e^{\lambda}\bar{Q}'_u$, $b = e^{\lambda}\bar{Q}'_v$ $\bar{Q}'_{uu} + \bar{Q}'_{vv} + 2e^{-2\lambda} = 0$
Middle Envelope a Point	$a = e^{-\lambda}R_v$ $b = -e^{-\lambda}R_u$		
Fundamental Quantities	$\xi_1 = \eta = p = q_1 = 0$ $\xi = \eta_1 = e^{\lambda}$ $p_1 = -e^{\lambda}$, $q = e^{\lambda}$ $r = -\lambda_v$, $r_1 = \lambda_u$	$\xi_1 = \eta = p = q_1 = 0$ $\xi = \eta_1 = e^{-\lambda}$ $p_1 = q = -e^{\lambda}$ $r = \lambda_v$, $r_1 = -\lambda_u$	$\xi_1 = \eta = p_1 = q = 0$ $\xi = \eta_1 = e^{-\lambda}$ $p = e^{\lambda}$, $q_1 = -e^{\lambda}$ $r = \lambda_v$, $r_1 = -\lambda_u$

17. **Some theorems concerning rotated congruences.** When a normal congruence is referred to the surface S_1,

$$a = e^{-\lambda} \bar{P}_u, \qquad b = -e^{-\lambda} \bar{P}_v;$$

the point in the tangent plane symmetrical with this with respect to the x-axis is

$$a_1 = e^{-\lambda} \bar{P}_u, \qquad b_1 = e^{-\lambda} \bar{P}_v.$$

Hence from the table of § 16, we have:

THEOREM 6. *If each line of a normal congruence referred to S_1 be reflected with respect to the xz-plane of the trihedral, the coördinates defining the reflected line when plotted relative to the trihedral on the sphere, define also a normal congruence. Similarly, the coördinates defining the reflected line with reference to the yz-plane, when plotted relative to the trihedral on the sphere, determine a normal congruence.*

From the table of § 16, we have at once the following two theorems:

THEOREM 7. *If each line of a normal congruence referred to S_1 be rotated through an angle $\pi/2$ about the corresponding normal, then when plotted relative to the trihedral of S_2, the coördinates defining the rotated congruence define also a normal congruence.*

THEOREM 8. *If each line of a normal congruence referred to S_2 be reflected with respect to either the xz- or the yz-plane, the coördinates defining the reflected congruence when plotted relative to the trihedral on the sphere, define a congruence whose middle envelope is a point.*

For an isotropic congruence referred to the sphere,

$$a = e^{\lambda} Q_u, \qquad b = -e^{\lambda} Q_v$$

$$Q_{uu} + Q_{vv} = 0;$$

and for a normal congruence referred to S_1,

$$a_1 = e^{-\lambda} \bar{P}_u, \qquad b_1 = -e^{-\lambda} \bar{P}_v,$$

where $\bar{P}$ is an arbitrary function of u and v. Then

$$\sqrt{a^2 + b^2} = e^{\lambda} (Q_u^2 + Q_v^2)^{\frac{1}{2}},$$

$$\sqrt{a_1^2 + b_1^2} = e^{-\lambda} (\bar{P}_u^2 + \bar{P}_v^2)^{\frac{1}{2}}.$$

Hence

$$\sqrt{a^2+b^2}\ \sqrt{a_1^2+b_1^2} = (Q_u^2+Q_v^2)^{\frac{1}{2}}\,(P_u^2+\bar{P}_v^2)^{\frac{1}{2}}.$$

Let us assume $\bar{P}$ to be a solution of Laplace's equation. We take

$$Q_u^2+Q_v^2 = U+iV, \tag{46}$$

$$\bar{P}_u^2+\bar{P}_v^2 = \frac{1}{U+iV}. \tag{47}$$

The modulus in (46) is $\sqrt{U^2+V^2}$, and the modulus in (47) is $1/\sqrt{U^2+V^2}$. Then

$$\sqrt{a^2+b^2}\ \sqrt{a_1^2+b_1^2} = 1,$$

and therefore the points (a, b) and (a_1, b_1) are inversions of each other with respect to the unit circle. Hence we have:

THEOREM 9. *If the point defining an isotropic congruence referred to the sphere be inverted with respect to the unit circle, the coördinates of the inverted point when plotted relative to the trihedral on S_1, define a normal congruence.*

For a normal congruence referred to S_2,

$$a = e^{-\lambda}\,\bar{P}_v', \qquad b = e^{-\lambda}\,\bar{P}_u',$$

and the equation defining the developables is

$$(\bar{P}_{uv}' - \lambda_u P_v' - \lambda_v \bar{P}_u' + 1)(du^2 - dv^2) + (\bar{P}_{vv}' - \bar{P}_{uu}' + 2\lambda_u \bar{P}_u' - 2\lambda_v \bar{P}_v')\,du\,dv = 0.$$

Let us suppose the developables are represented by the asymptotic lines. Then

$$\bar{P}_{uv}' - \lambda_u \bar{P}_v' - \lambda_v \bar{P}_u' + 1 = 0. \tag{48}$$

If P_1 and P_2 are two solutions of this equation, we get by subtraction

$$(P_1 - P_2)_{uv} - \lambda_u (P_1 - P_2)_v - \lambda_v (P_1 - P_2)_u = 0.$$

Hence $P = P_1 - P_2$ is a solution of the identical equations (20) and (36). Consequently we have:

THEOREM 10. *If P_1 and P_2 are two solutions of* (48), *then for $P = P_1 - P_2$, the lines of curvature on the normal surfaces of the congruence referred to the sphere, and the lines of curvature on the normal surfaces of the congruence referred to S_1, have the same spherical representation, namely, the given isothermal system on the sphere.*

D. Ribaucour congruences.

17. The lines drawn from the points of one surface parallel to the normals at the corresponding points of a second surface which corresponds to the former by orthogonality of linear elements, is called a congruence of Ribaucour. The second surface is called the director surface.

We consider the director surface S referred to any orthogonal system, and choose the x-axis of the trihedral tangent to the curve $v = \text{const.}$ The lines l, drawn parallel to the normals of S, will constitute a congruence of Ribaucour when the locus of some point (a, b, t) on l, is a surface corresponding to S with orthogonality of linear elements. We shall denote the absolute displacements of $P(a, b, t)$ by $\delta x, \delta y, \delta z$, and the absolute displacements of the vertex of the trihedral by $\delta\bar{x}, \delta\bar{y}, \delta\bar{z}$. Then the condition that S and the locus of P be surfaces corresponding with orthogonality of linear elements, is that

$$\delta x\, \delta\bar{x} + \delta y\, \delta\bar{y} + \delta z\, \delta\bar{z} = 0, \tag{49}$$

for all values of dv/du. Since $\delta\bar{z} = 0$, (49) reduces to

$$\xi\, du\,(a_u\, du + a_v\, dv + \xi\, du + q\, t\, du + q_1\, t\, dv - r\, b\, du - r_1\, b\, dv)$$

$$+ \eta_1\, dv\,(b_u\, du + b_v\, dv + \eta_1\, dv + r\, a\, du + r_1\, a\, dv - p\, t\, du - p_1\, t\, dv) = 0.$$

Setting the coefficients of du^2, $du\,dv$, and dv^2 equal to zero, we get the following conditions for a Ribaucour congruence:

$$\begin{aligned} a_u + \xi + tq - br &= 0, \\ \xi(a_v + tq_1 - br_1) + \eta_1(b_u + ar - tp) &= 0, \\ b_v + \eta_1 + ar_1 - tp_1 &= 0. \end{aligned} \tag{50}$$

18. The following equations are readily derived by the methods outlined in section A.

The equation for the developables becomes after some reduction,

$$q\xi\,du^2 - 2p\eta_1\,du\,dv - p_1\eta_1\,dv^2 = 0, \tag{51}$$

which is the equation of the asymptotic lines; hence we have the well known theorem that the developables of a congruence of Ribaucour correspond to the asymptotic lines on the director surface.*

The equation for the distances to the focal points is

$$\begin{aligned}(pq_1 - p_1q)\,t^2 + [q(b_v + \eta_1 + ar_1) - p_1(a_u + \xi - br) \\ + p(a_v - br_1) - q_1(b_u + ar)]\,t \\ + (a_u + \xi - br)(b_v + \eta_1 + ar_1) - (a_v - br_1)(b_u + ar) = 0.\end{aligned} \tag{52}$$

The expression for t derived from the three equations (50), and the expression for one half the sum of the roots of (52), are identical; consequently, we have the well known theorem that the middle surface of a congruence of Ribaucour is the surface corresponding to the director surface with orthogonality of linear elements.†

If on the director surface we choose the lines of curvature parametric, the equation of the principal surfaces becomes, on using (50),

$$q^2\,du^2 - p_1^2\,dv^2 = 0. \tag{53}$$

Hence we have the theorem:

The curves on the director surface defining the principal surfaces of a congruence of Ribaucour are symmetrically placed with reference to the lines of curvature.

Since the asymptotic lines define the developables, and since the focal planes are symmetrically placed with reference to the principal planes, the asymptotic lines are also symmetrically placed with reference to the lines of curvature. The spherical representation of the asymptotic lines is orthogonal to their direction on the director surface; hence the spherical representation of the asymptotic lines is symmetrically placed with reference to the spherical representation of the lines of curvature.

The equation for the distances to the limit points is

$$\begin{aligned}4p_1^2q^2\,t^2 + 4p_1q[p_1(a_u + \xi - br) - q(b_v + \eta_1 + ar_1)]\,t \\ - 4p_1q(a_u + \xi - br)(b_v + \eta_1 + ar_1) + [q(a_v - br_1) - p(b_u + ar)]^2 = 0.\end{aligned} \tag{54}$$

* Eisenhart, p. 422.

† Eisenhart, p. 421.

19. We have seen that the developables of a congruence of Ribaucour correspond to the asymptotic lines on the director surface. When two surfaces correspond with orthogonality of linear elements, the asymptotic lines on one correspond to a conjugate system on the other.* Hence, since the middle surface corresponds to the director surface with orthogonality of linear elements, we have the well known theorem:

The developable surfaces of a congruence of Ribaucour cut the middle surface in a conjugate system.†

Guichard has proved‡ that a congruence with this property is a congruence of Ribaucour.

20. **Ribaucour congruence whose director surface is the sphere.** We refer the sphere to any isothermal system. Equations (50) become

$$\begin{aligned} a_u + e^{\lambda} + t e^{\lambda} + b\lambda_v &= 0, \\ a_v + b_u - a\lambda_v - b\lambda_u &= 0, \\ b_v + e^{\lambda} + t e^{\lambda} + a\lambda_u &= 0. \end{aligned} \tag{55}$$

From the second equation, we define a function Q by the equations,

$$\frac{\partial}{\partial v}(a e^{-\lambda}) = -\frac{\partial}{\partial u}(b e^{-\lambda}) = Q_{uv}. \tag{56}$$

From the first and third equations of (55) we get by subtraction

$$a_u - a\lambda_u - b_v + b\lambda_v = 0.$$

This gives

$$\frac{\partial}{\partial u}(a e^{-\lambda}) = \frac{\partial}{\partial v}(b e^{-\lambda}). \tag{57}$$

Hence from (56) and (57),

$$\begin{aligned} a &= e^{\lambda} Q_u, \qquad b = -e^{\lambda} Q_v, \\ Q_{uu} &+ Q_{vv} = 0. \end{aligned} \tag{58}$$

* Eisenhart, p. 384.

† Eisenhart, p. 421.

‡ Annales L'Ecole Normale, Ser. 3, vol. VI (1889), pp. 344, 345.

Since this condition is identical with the condition that a congruence referred to the sphere be isotropic, we have the well known theorems:

When the director surface of a congruence of Ribaucour is a sphere, the congruence is isotropic.*

The middle surface of an isotropic congruence corresponds to the sphere with orthogonality of linear elements.*

21. **Ribaucour congruences whose director surface is a minimal surface.** We choose the lines of curvature parametric. Equations (50) become

$$\begin{aligned} &a_u + e^{-\lambda} - t\,e^{\lambda} - b\,\lambda_v = 0, \\ &a_v + b_u + a\,\lambda_v + b\,\lambda_u = 0, \\ &b_v + e^{-\lambda} - a\,\lambda_u + t\,e^{\lambda} = 0. \end{aligned} \tag{59}$$

From the second equation, we define a function P by the equations,

$$\frac{\partial}{\partial v}(a\,e^{\lambda}) = -\frac{\partial}{\partial u}(b\,e^{\lambda}) = P_{uv}. \tag{60}$$

From the first and third equations of (59), we get by addition

$$a_u - a\,\lambda_u + b_v - b\,\lambda_v + 2e^{-\lambda} = 0. \tag{61}$$

From (60),

$$a = e^{-\lambda} P_u, \qquad b = -e^{-\lambda} P_v; \tag{62}$$

these values of a and b, when substituted in (61), give

$$P_{uu} - P_{vv} - 2\,\lambda_u P_u + 2\,\lambda_v P_v + 2 = 0. \tag{63}$$

Hence equations (62) and (63) define a and b for a Ribaucour congruence, when the director surface is minimal. Since (62) is identical with the condition that a congruence be normal when referred to a minimal surface whose lines of curvature are parametric, we have the theorem:†

When the director surface of a congruence of Ribaucour is minimal, the congruence is normal.

* Eisenhart, Ex. 12, p. 424.

† Eisenhart, p. 422.

22. We have seen that when the asymptotic lines on a minimal surface are parametric, the condition that a congruence be normal is that

$$a = e^{-\lambda}\bar{P}'_v, \qquad b = e^{-\lambda}\bar{P}'_u. \tag{64}$$

Now $\bar{P}'$ is any function whatever of u and v; let us suppose $\bar{P}'$ satisfies (63). Then we have:

THEOREM 11. *If the point defined by* (63) *and* (64) *be plotted relative to the trihedral on S_1, the adjoint minimal surface, and then rotated negatively through an angle $\pi/2$ about the normal, the rotated point will have the coördinates*

$$a_1 = e^{-\lambda}\bar{P}'_u, \qquad b_1 = -e^{-\lambda}\bar{P}'_v,$$

and therefore in consequence of (62), *defines a congruence of Ribaucour. Conversely, every Ribaucour congruence whose director surface is minimal is derived in this manner.*

YALE UNIVERSITY,
June, 1923.

PARALLEL PROPAGATION OF A VECTOR AROUND AN INFINITESIMAL CIRCUIT IN AN AFFINE-CONNECTED MANIFOLD.

By J. L. Synge.

1. Being given the components of affine connection $\Gamma^i_{jk} (= \Gamma^i_{kj})$ as functions of coördinates in a manifold V_N, with a coördinate system x_i, a vector ξ^i is said to be propagated parallelly along a curve $x_i = x_i(u)$ if the components satisfy

$$\frac{d\xi^i}{du} + \Gamma^i_{jk}\,\xi^j \frac{dx_k}{du} = 0 \tag{1}$$

along the curve. The derivative notation is deliberately adopted instead of the differential, as the use of the latter has caused an inexactitude of reasoning on the part of certain writers. We find one of the most interesting problems of parallel propagation in the case where the curve of propagation is closed, or, in particular, an "infinitesimal circuit", but it is necessary to employ more than the customary precision in the use of the latter phrase.

2. The problem of a finite circuit is, according to Weyl,* reducible to that of the infinitesimal circuit. No proof of this statement is given, and indeed it does not appear to be true except in the case of a "Euclidean-affine" manifold. A generalised Stokes' Theorem is certainly not applicable, as Eddington suggests.† Weyl treats the problem of the infinitesimal circuit and obtains a result which may be stated as follows:

If a vector ξ^i is propagated parallelly around the infinitesimal parallelogram x_i, $x_i + dx_i$, $x_i + dx_i + \delta x_i$, $x_i + \delta x_i$, x_i, in the order indicated, the increments of the components of the vector on completion of the circuit are given by

$$\Delta\xi^i = -F^i_{jkl}\,\xi^j\,dx_k\,\delta x_l \tag{2}$$

where

$$F^i_{jkl} = A^i_{jkl} - A^i_{jlk}, \qquad A^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x_k} - \Gamma^i_{rl}\,\Gamma^r_{jk}.$$

The same result is obtained by Laue‡ for a Riemannian manifold, but his method is equally applicable to the affine case. However in spite of Weyl's

* Raum, Zeit, Materie, 4th ed., p. 106.

† Espace, Temps et Gravitation, (1921), Partie théorique, p. 53.

‡ Relativitätstheorie, Bd. II, p. 93.

assertion that his differential treatment may be rigorised by the substitution of derivatives for differentials, we shall see later that a rigorous discussion of the problem demands the employment of additional second derivatives: in the omission of these both he and Laue are guilty of error, while obtaining the correct result.

Eddington (loc. cit.) takes a covariant vector in Riemannian space—presumably given as a function of the coördinates, since the partial derivatives of the vector with respect to the coördinates occur—and proceeds to find the variation on propagation around an infinitesimal circuit. He obtains a correct formal result, but should equate it to zero, since he assumed the integrability conditions (i. e., the vanishing of the curvature tensor) when he took the vector as a function of position. Later* he treats the same question for a more general geometry but still one apparently with a fundamental tensor g_{mn}, permitting the conversion of a contra— to a covariant tensor.

The looseness in the processes described above has led M. P. Dienes† to state that Weyl's result (2) is not exact. By use of (1), in its differential form, he obtains results in discord with (2) for special cases. These results he characterises as absurd and states: "The reason for all these difficulties is that one claims to obtain a precision of the second degree from an approximation of the first degree." In fact he rejects (1) as incompetent to obtain results, and substitutes for it a new law of propagation—his equation (6). But the meaning of this new law is not clear, for if dx_i represent actual finite increments of the coördinates, his law is not invariant, while if dx_i are differentials his equation is meaningless, as the right hand side is not homogeneous, as every equation in differentials must be. It would appear that the cause of his anomalous results is to be found in a loose employment of the differential notation, in which he has been less fortunate than Weyl.

3. We shall enunciate and prove the theorem concerning propagation around an infinitesimal circuit.

THEOREM: *In an affine-connected manifold a point P is taken, and through it there passes a two-space given by*

$$x_i = x_i(u, v),$$

so chosen that P is an ordinary point, i. e., so that the partial derivatives of x_i with respect to u and v up to the third order exist and are continuous at and near P. A contour C in the two-space is formed by the parametric lines

* Loc. cit., p. 125.

† Comptes Rendus, t. 174, No. 18, p. 1167.

of u and v joining $P(u,v)$, $Q(u+\Delta u, v)$, $R(u+\Delta u, v+\Delta v)$, $S(u, v+\Delta v)$ and P in order, Δu and Δv being finite. If a vector ξ^i be propagated parallelly around C in the above order, starting with the components $(\xi^i)_P$ and ending with $(\bar{\xi}^i)_P$, and if

$$(\Delta \xi^i)_P = (\bar{\xi}^i)_P - (\xi^i)_P,$$

then

$$\lim_{\Delta u, \Delta v \to 0} \frac{(\Delta \xi^i)_P}{\Delta u \, \Delta v} = -F^i_{jkl} \xi^j \frac{\partial x_k}{\partial u} \frac{\partial x_l}{\partial v}$$

where Δu and Δv approach zero simultaneously in such manner that their ratio has a finite limit other than zero.

Employing ϵ as a general symbol for any quantity such that

$$\lim_{\Delta u, \Delta v \to 0} \frac{\epsilon}{\Delta u \, \Delta v} = 0,$$

we have the following equations corresponding respectively to the lines PQ, QR, RS, SP:

$$(\xi^i)_Q = (\xi^i)_P + \Delta u \left(\frac{d\xi^i}{du}\right)_P + \frac{(\Delta u)^2}{2} \left(\frac{d^2 \xi^i}{du^2}\right)_P + \epsilon$$

$$(\xi^i)_R = (\xi^i)_Q + \Delta v \left(\frac{d\xi^i}{dv}\right)_Q + \frac{(\Delta v)^2}{2} \left(\frac{d^2 \xi^i}{dv^2}\right)_Q + \epsilon$$

$$(\xi^i)_S = (\xi^i)_R - \Delta u \left(\frac{d\xi^i}{du}\right)_R + \frac{(\Delta u)^2}{2} \left(\frac{d^2 \xi^i}{du^2}\right)_R + \epsilon$$

$$(\xi^i)_P = (\xi^i)_S - \Delta v \left(\frac{d\xi^i}{dv}\right)_S + \frac{(\Delta v)^2}{2} \left(\frac{d^2 \xi^i}{dv^2}\right)_S + \epsilon.$$

By addition we have

$$(\Delta \xi^i)_P = U_1 + V_1 + U_2 + V_2 + \epsilon,$$

where

$$U_1 = \Delta u \left\{ \left(\frac{d\xi^i}{du}\right)_P - \left(\frac{d\xi^i}{du}\right)_R \right\} \qquad V_1 = \Delta v \left\{ \left(\frac{d\xi^i}{dv}\right)_Q - \left(\frac{d\xi^i}{dv}\right)_S \right\}$$

$$U_2 = \frac{1}{2} (\Delta u)^2 \left\{ \left(\frac{d^2 \xi^i}{du^2}\right)_P + \left(\frac{d^2 \xi^i}{du^2}\right)_R \right\} \qquad V_2 = \frac{1}{2} (\Delta v)^2 \left\{ \left(\frac{d^2 \xi^i}{dv^2}\right)_Q + \left(\frac{d^2 \xi^i}{dv^2}\right)_S \right\}.$$

The calculation of these quantities by means of (1) gives

$$U_1 = \left(A^i_{jlk} \xi^j \frac{\partial x_k}{\partial u} \frac{\partial x_l}{\partial u} + \Gamma^i_{jk} \xi^j \frac{\partial^2 x_k}{\partial u^2} \right)_P (\Delta u)^2$$

$$+ \left(A^i_{jlk} \xi^j \frac{\partial x_k}{\partial u} \frac{\partial x_l}{\partial v} + \Gamma^i_{jk} \xi^j \frac{\partial^2 x_k}{\partial u \, \partial v} \right)_P \Delta u \, \Delta v + \epsilon$$

$$V_1 = -\left(A^i_{jkl}\,\xi^j\,\frac{\partial x_k}{\partial u}\,\frac{\partial x_l}{\partial v} + \Gamma^i_{jk}\,\xi^j\,\frac{\partial^2 x_k}{\partial u\,\partial v}\right)_P \Delta u\,\Delta v$$
$$+\left(A^i_{jlk}\,\xi^j\,\frac{\partial x_k}{\partial v}\,\frac{\partial x_l}{\partial v} + \Gamma^i_{jk}\,\xi^j\,\frac{\partial^2 x_k}{\partial v^2}\right)_P (\Delta v)^2 + \epsilon$$

$$V_2 = -\left(A^i_{jlk}\,\xi^j\,\frac{\partial x_k}{\partial u}\,\frac{\partial x_l}{\partial u} + \Gamma^i_{jk}\,\xi^j\,\frac{\partial^2 x_k}{\partial u^2}\right)_P (\Delta u)^2 + \epsilon$$

$$V_3 = -\left(A^i_{jlk}\,\xi^j\,\frac{\partial x_k}{\partial v}\,\frac{\partial x_l}{\partial v} + \Gamma^i_{jk}\,\xi^j\,\frac{\partial^2 x_k}{\partial v^2}\right)_P (\Delta v)^2 + \epsilon.$$

Addition gives

$$(\Delta\xi^i)_P = \left[(A^i_{jlk} - A^i_{jkl})\,\xi^j\,\frac{\partial x_k}{\partial u}\,\frac{\partial x_l}{\partial v}\right]_P \Delta u\,\Delta v + \epsilon$$
$$= -\left(F^i_{jkl}\,\xi^j\,\frac{\partial x_k}{\partial u}\,\frac{\partial x_l}{\partial v}\right)_P \Delta u\,\Delta v + \epsilon,$$

whence the result follows directly.

4. Having thus assigned a geometrical meaning to the quantities

$$\eta^i = -F^i_{jkl}\,\xi^j\,\frac{\partial x_k}{\partial u}\,\frac{\partial x_l}{\partial v}, \qquad (i = 1, 2, \ldots, N), \tag{3}$$

it is to be observed that while their magnitudes are dependent on the choice of the surface coördinates, the ratios $\eta^1 : \eta^2 : \ldots : \eta^N$ are not so dependent. For taking different surface coördinates u', v' and writing

$$\eta'^i = -F^i_{jkl}\,\xi^j\,\frac{\partial x_k}{\partial u'}\,\frac{\partial x_l}{\partial v'},$$

we have

$$\eta'^i = -F^i_{jkl}\,\xi^j\left(\frac{\partial x_k}{\partial u}\,\frac{\partial u}{\partial u'} + \frac{\partial x_k}{\partial v}\,\frac{\partial v}{\partial u'}\right)\left(\frac{\partial x_l}{\partial u}\,\frac{\partial u}{\partial v'} + \frac{\partial x_l}{\partial v}\,\frac{\partial v}{\partial v'}\right),$$

which, since F^i_{jkl} is skew symmetric in k and l, becomes

$$\eta'^i = \frac{\partial(u, v)}{\partial(u', v')}\,\eta^i.$$

Thus we have the general result:

Being given a vector and a two-space element at a point, there is defined (under certain restrictions) a definite direction at the point, given by (3).

We must observe that the foregoing method is not adequate to discuss the propagation around an infinitesimal circuit on a cone, whose vertex lies at the point under consideration.

ON THE METHOD OF LEAST mTH POWERS FOR A SET OF SIMULTANEOUS EQUATIONS.*

By Dunham Jackson.

1. Introduction. Let

$$(1)\qquad \begin{array}{c} a_{11},\ a_{12},\ \ldots,\ a_{1p}, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1},\ a_{n2},\ \ldots,\ a_{np} \end{array}$$

be a set of real quantities, the rank of whose matrix is n. That is, $p \geq n$, and the a's form n linearly independent sets of p quantities each. Let

$$b_1,\ b_2,\ \ldots,\ b_p$$

be another set of p real quantities. The p simultaneous linear equations in $x_1, \cdots, x_n$:

$$(2)\qquad \begin{array}{l} a_{11}\,x_1 + \cdots + a_{n1}\,x_n = b_1, \\ a_{12}\,x_1 + \cdots + a_{n2}\,x_n = b_2, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ a_{1p}\,x_1 + \cdots + a_{np}\,x_n = b_p, \end{array}$$

will have a solution if $p = n$, but will generally have no solution if $p > n$. In the latter case, the question may be raised of determining $x_1, \cdots, x_n$, so that the sum

$$\sum_{i=1}^{p} |\, a_{1i}\,x_1 + \cdots + a_{ni}\,x_n - b_i \,|^m,$$

for a given exponent m, shall be a minimum.

For $m = 2$, this is the classical problem of least squares. For a general m, it is intimately related to a problem which has been considered by the writer in a recent series of papers† on the approximate representation of

* Presented to the American Mathematical Society, December 30, 1920.

† A. On functions of closest approximation, Trans. Amer. Math. Soc., vol. 22 (1921), pp. 117–128; B. Note on a class of polynomials of approximation, ibid., vol. 22 (1921), pp. 320–326; C. On the convergence of certain trigonometric and polynomial approximations, ibid., vol. 22 (1921), pp. 158–166; D. Note on an ambiguous case of approximation, ibid., vol. 25 (1923), pp. 333–337.

a given continuous function. The function b_i of the subscript i takes the place of the function $f(x)$ of a continuous variable x, for which an approximate representation is sought, and the sets of numbers a_{ki}, $k = 1, 2, \ldots, n$, regarded as constituting n linearly independent functions of i, correspond to the n linearly independent functions of x in terms of which the approximation is to be expressed. While the two cases, continuous and discrete, can be handled to a large extent along the same general lines, there does not seem to be so close an identity in detail as to render a separate treatment of the algebraic problem superfluous; and it is to a study of the latter problem that the present paper is devoted.

2. **Existence of a solution of the problem of closest approximation.*** $m > 0$. Let

$$\varphi_i = a_{1i} x_1 + \cdots + a_{ni} x_n, \qquad i = 1, 2, \cdots, p, \tag{3}$$

and let H be the greatest of the p numbers $|\varphi_i|$, for any given set of x's. We begin by establishing the lemma:

There exists a number P, completely determined by the matrix of the np numbers a_{ki}, such that

$$|x_k| \leq PH, \qquad k = 1, 2, \ldots, n,$$

however the numbers x_k may be chosen.

The order of the columns in the matrix (1) is immaterial, provided, of course, that any permutation of them is accompanied by a corresponding permutation of the b's; let it be supposed for definiteness that the determinant δ of the first n columns is different from zero. Let δ_{ki} be the cofactor of a_{ki} in the determinant δ; then, from the first n of the equations (3),

$$x_k = \frac{1}{\delta}(\delta_{k1}\varphi_1 + \delta_{k2}\varphi_2 + \cdots + \delta_{kn}\varphi_n), \qquad k = 1, 2, \ldots, n.$$

Let δ' be the greatest of the quantities $|\delta_{ki}|$; then, since $|\varphi_i| \leq H$,

$$|x_k| \leq nH\delta'/\delta,$$

and it suffices to take

$$P = n\delta'/\delta.$$

* It is readily seen that the problem becomes trivial for $m \leq 0$. The ground of this section is covered to some extent by F. Riesz, Les systèmes d'équations linéaires à une infinité d'inconnues, Paris, 1913, Chapter III, see especially pp. 51, 54; but the purposes and methods of Riesz's discussion are altogether different, and he does not formulate the conclusions presented here.

This proof is essentially the same as that given by Sibirani* for the continuous case. It would be possible, though not so simple, to give a demonstration corresponding more closely to that of Lemma I in the paper A.

For any exponent $m > 0$ and any given set of x's, let

$$G_m = \sum_{i=1}^{p} |\varphi_i|^m.$$

Since each term of the sum is positive or zero, no term can be greater than the whole sum, and

$$|\varphi_i|^m \leq G_m, \qquad |\varphi_i| \leq G_m^{1/m}, \qquad i = 1, 2, \ldots, p,$$

that is,

$$H \leq G_m^{1/m}.$$

Consequently, as a result of the preceding lemma, it is seen that:

However the numbers x_k may be chosen,

$$|x_k| \leq PG_m^{1/m}, \qquad k = 1, 2, \ldots, n,$$

where P has the same meaning as before.

Now let a set of numbers $b_1, \ldots, b_p$ be given, or, as we shall say for brevity, a *vector* $(b_1, \ldots, b_p) = (b)$. Let

$$g_m = \sum_{i=1}^{p} |\varphi_i - b_i|^m.$$

It is to be shown that the coefficients x_k in the φ's can be determined so that the value of g_m shall reach its lower limit, the exponent m and the numbers a_{ki} and b_i being held fast.

If the vector (b) is linearly dependent on the vectors $(a_k) = (a_{k1}, \ldots, a_{kp})$, $k = 1, 2, \ldots, n$, it is possible to choose the x's so that $\varphi_i = b_i$, $i = 1, 2, \ldots, p$, and g_m is reduced to its lower limit, zero. This means that the set of equations (2) has an exact solution.

If (b) is not linearly dependent on the vectors (a_k), the a's and the b's together define a set of $n+1$ linearly independent vectors. The numbers

* Sulla rappresentazione approssimata delle funzioni, Annali di mat., ser. 3, vol. 16 (1909), pp. 203–221; p. 208; see also Tonelli, I polinomi d'approssimazione di Tchebychef, ibid., vol. 15 (1908), pp. 47–119; pp. 61–62.

$(\varphi_1 - b_1, \ldots, \varphi_p - b_p)$ result from a linear combination of these, with the coefficients $x_1, \ldots x_n, -1$. If Q is, for the augmented system of vectors, the coefficient corresponding to the number P already associated with the original system, it will follow that

$$|x_k| \leq Q g_m^{1/m}, \qquad k = 1, 2, \ldots, n;$$

the number Q, then, depends only on the a's and the b's.

The last relation implies that if g_m is near its lower limit, the point in n-dimensional space which has the numbers x_k for its coördinates belongs to a restricted region, which may be regarded as closed, and when g_m is considered as a function of the x's, its lower limit is a minimum which can be attained by a suitable choice of the variables. Consequently, whether (b) is linearly dependent on the vectors (a_k) or not,

There will exist at least one determination of $x_1, \ldots, x_n$, for which the sum of the mth powers of the absolute values of the errors in the equations (2) *is a minimum.*

3. **Uniqueness of the solution,** $m > 1$. It remains to inquire whether the solution, just proved to exist, is unique. It is readily seen that when $m < 1$ this is not necessarily the case. For example, let the given quantities be taken as follows:

$$p = 2, \qquad n = 1, \qquad a_{11} = a_{12} = 1, \qquad b_1 = 1, \qquad b_2 = -1.$$

The expression to be minimized is

$$g_m = |x_1 - 1|^m + |x_1 + 1|^m.$$

As g_m is an even function of x_1, the determination of x_1 for the minimum can not be unique, unless it is the value $x_1 = 0$; but $g_m = 2$ for $x_1 = 0$, while $g_m = 2^m$ for $x_1 = \pm 1$, so that the former value is not the minimum. The solution is in fact given by $x_1 = \pm 1$.

If $m = 1$, the preceding example can still be used to show that the solution need not be unique.* The minimum in this case, $g_1 = 2$, is reached for any value of x_1 in the interval $-1 \leq x_1 < 1$.

For $m > 1$, on the other hand, the solution of the problem of closest approximation is uniquely defined. Let the smallest value of g_m be denoted by γ_m. Let $(x_1', \ldots, x_n')$ and $(x_1'', \ldots, x_n'')$ be two different sets of x's, if possible, which make $g_m = \gamma_m$. Let φ_i' and φ_i'' be the corresponding values

* This is an essential difference between the discrete case and the most important, at any rate, of the continuous cases; cf. the paper B.

of φ_i. Because of the linear independence of the a's, φ_i' and φ_i'' can not be equal for all values of i. Let h be a subscript such that $\varphi_h' \neq \varphi_h''$. Then

$$\begin{aligned} |\tfrac{1}{2}(\varphi_h' + \varphi_h'') - b_h|^m &= |\tfrac{1}{2}(\varphi_h' - b_h) + \tfrac{1}{2}(\varphi_h'' - b_h)|^m \\ &< \tfrac{1}{2}(|\varphi_h' - b_h|^m + |\varphi_h'' - b_h|^m). \end{aligned}$$

This is an immediate consequence of the fact that the chord connecting any two points of the curve $\eta = |\xi|^m$ lies above the curve, the mth power of the absolute value of the average of two numbers and the average of the mth powers of the absolute values of the same numbers being represented respectively by a point on the curve and a point with the same abscissa on a chord. (It is here that the demonstration fails for $m \leqq 1$.) For any other subscript i, the same relations hold, except that the inequality becomes an equality whenever $\varphi_i' = \varphi_i''$. At any rate, since the inequality holds at least once,

$$\sum_{i=1}^{p} |\tfrac{1}{2}(\varphi_i' + \varphi_i'') - b_i|^m < \gamma_m,$$

the quantity obtained on the right, on summation from $i = 1$ to $i = p$, reducing to the value indicated. But the left-hand member in the last inequality is a value of g_m, namely that given by $x_k = \frac{1}{2}(x_k' + x_k'')$, $k = 1, 2, \ldots, n$, and the hypothesis with regard to γ_m is thereby contradicted.

4. **Approximate solution in the sense of Tchebychef.** A problem closely related to the one treated above, and suggested by the Tchebychef theory of approximation by means of polynomials,* is that of determining the x's so that the greatest of the errors $|\varphi_i - b_i|$ in the equations (2) shall be as small as possible. It follows almost immediately from the first lemma of § 2 that *the problem has at least one solution*; for the x's corresponding to restricted values of the errors will themselves belong to a restricted region, and the lower limit involved will be a minimum that can be attained.

Under the hypotheses made hitherto, the solution of the new problem is not necessarily unique. For example, let the given quantities be as in the illustration at the beginning of § 3, except that $a_{12} = 0$ instead of 1. The problem is to determine x_1 so that the greater of the quantities

$$|\varphi_1 - b_1| = |x_1 - 1|, \qquad |\varphi_2 - b_2| = 1,$$

* Cf., e. g., Sibirani, Tonelli, locc. citt.; Borel, Leçons sur les fonctions de variables réelles et les développements en séries de polynomes, Paris, 1905, pp. 82–92; de la Vallée Poussin, Leçons sur l'approximation des fonctions d'une variable réelle, Paris, 1919, pp. 74–92.

shall be as small as possible, and the requirement is satisfied by any value of x_1 in the interval $0 < x_1 \leqq 2$.

The solution will be completely determinate, if it is required that every n-rowed determinant of the matrix (1) *be different from zero**. The reasoning corresponds closely to that which is well known in the theory of Tchebychef polynomials.

In the first place, if $x_1, \ldots, x_n$ form a solution of the problem, if $q_1, \ldots, q_p$ have the values corresponding to these x's, and if l is the greatest of the errors $|q_i - b_i|$, it must be that

$$|q_i - b_i| = l$$

for at least $n+1$ different values of the subscript i. Suppose the maximum deviation were reached only r times, $r \leqq n$. Since the original order of subscripts is immaterial, there is no loss of generality in supposing this to occur for $i = 1, 2, \ldots, r$. That is,

$$\begin{aligned} |q_i - b_i| &= l, & i &= 1, \ldots, r, \\ |q_i - b_i| &< l, & i &= r+1, \ldots, p. \end{aligned}$$

As every n-rowed determinant of the matrix (1), and, in particular, the determinant of the first n columns, is assumed now to be different from zero, there will exist a set of numbers $y_1, \ldots, y_n$, satisfying the equations

$$a_{1i} y_1 + \cdots + a_{ni} y_n = q_i - b_i, \qquad i = 1, 2, \ldots, n.$$

The y's having been determined, let

$$\psi_i = a_{1i} y_1 + \cdots + a_{ni} y_n$$

for $i = 1, 2, \ldots, p$. Let ϵ be a positive number less than 1, and at the same time subject to the inequalities

$$\epsilon |\psi_i| < l - |q_i - b_i|, \qquad i = n+1, \ldots, p.$$

If the quantities

$$x_k - \epsilon y_k, \qquad k = 1, 2, \ldots, n,$$

are substituted for the unknowns in the equations (2), the resulting errors

$$|q_i - \epsilon \psi_i - b_i|$$

will be less than l in every case, and the hypothesis will be contradicted.

* Cf. Sibirani, Su la rappresentazione approssimata di una funzione continua, etc., Rend. Circ. Mat. Palermo, vol. 34 (1912), pp. 132–157.

Suppose now that there were two sets of numbers, $(x_1', \ldots, x_n')$ and $(x_1'', \ldots, x_n'')$, each giving a solution of the minimum problem; and let q_i' and q_i'', $i = 1, 2, \ldots, p$, be the corresponding results of substitution in the left-hand members of the equations (2). For any particular subscript i, unless $q_i' = q_i'' = b_i \pm l$, the quantity $\frac{1}{2}(q_i' + q_i'')$, corresponding to the substitution of $\frac{1}{2}(x_k' + x_k'')$, $k = 1, 2, \ldots, n$, will differ from b_i by less than l. As the difference can not exceed l in any case, the new values for the unknowns will also constitute a solution of the problem; by the preceding paragraph, it must be that $|\frac{1}{2}(q_i' + q_i'') - b_i| = l$ for at least $n+1$ values of i; and hence it must be that $q_i' = q_i''$ for at least $n+1$ values of i. But the equality of q_i' and q_i'', even for n values of i, implies that $x_k' = x_k''$, $k = 1, 2, \ldots, n$. So the proof of uniqueness is complete.

5. **Limiting form of the problem of least mth powers for $m = \infty$.** Let $x_{m1}, \ldots, x_{mn}$ be the x's giving the solution of the problem of least mth powers, for an arbitrary value of $m > 1$; let q_{mi}, $i = 1, 2, \ldots, p$, be the corresponding value of the left-hand member of the ith of the equations (2); and let l_m be the greatest of the differences $|q_{mi} - b_i|$. Let l_0 be the limiting error in the problem of the preceding section, regardless, for the moment, whether the solution of that problem is unique or not, the hypothesis with regard to the matrix (1) being merely that it is of rank n; and let $x_{01}, \ldots, x_{0n}$ and $q_{01}, \ldots, q_{0p}$ be the corresponding set, or a corresponding set, of x's and q's. *It will be shown that*

$$\lim_{m=\infty} l_m = l_0.$$

Let ε be an arbitrarily small positive quantity, let γ_m, as before, stand for the sum $\sum |q_{mi} - b_i|^m$, and suppose that

$$l_m \geq l_0 + \varepsilon.$$

Then at least one term in the expression for γ_m is as great as $(l_0 + \varepsilon)^m$, and as the other terms are positive or zero,

$$\gamma_m \geq (l_0 + \varepsilon)^m.$$

On the other hand, because of the minimum property of γ_m, its value can not exceed that of the corresponding sum obtained by the substitution of any other set of x's, in particular $x_{01}, \ldots, x_{0n}$. But

$$|q_{0i} - b_i| \leq l_0, \qquad i = 1, 2, \ldots, p,$$

and consequently

$$\sum_{i=1}^{p} | \varphi_{0i} - b_i |^m \leq p l_0^m,$$

$$\gamma_m \leq p l_0^m, \qquad (l_0 + \epsilon)^m \leq p l_0^m.$$

As the last relation ceases to be true for large values of m, it must be that

$$l_m < l_0 + \epsilon$$

for all values of m from a certain point on, that is, $\lim_{m=\infty} l_m = l_0$.

Now let it be assumed that the determination of the numbers $x_{01}, \ldots, x_{0n}$ is unique. From the preceding section, this will certainly be the case if the n-rowed determinants of the matrix (1) are all different from zero. By the proof just completed, l_m is bounded as m becomes infinite. That is, the quantities $| \varphi_{mi} - b_i |$ are bounded, and hence the same thing is true of the quantities $| \varphi_{mi} |$. From this it follows, by the first lemma of § 2, that the points in n-dimensional space represented by the coördinates $(x_{m1}, \ldots, x_{mn})$ belong to a restricted region as m becomes infinite, and so have at least one limit-point* for $m = \infty$. On the other hand, if l represents the greatest of the p numbers $| \varphi_i - b_i |$ for an arbitrary set of x's, this l is a continuous function of the x's, and the value of l corresponding to any limit-point of a sequence of points $(x_{m1}, \ldots, x_{mn})$ for $m = \infty$ must be the limiting value of l_m, namely l_0. But, under the present hypothesis, the value l_0 can correspond only to the single point $(x_{01}, \ldots, x_{0n})$. So this is the only possible limit-point for $m = \infty$, and *it must be that*

$$\lim_{m=\infty} x_{mk} = x_{0k}, \qquad k = 1, 2, \ldots, n.$$

* That is, there will be a sequence of values of m, increasing without limit, such that the corresponding points $(x_{m1}, \ldots, x_{mn})$ approach the point in question as a limit.

The University of Minnesota,
Minneapolis, Minn.

THE DERIVATIVE OF THE GENERAL INTEGRAL.

By P. J. Daniell.

1. The object of this paper is to consider a function which may be compared to a derivative, and which is related to the general form of integral already defined by the author.*

In that definition it was assumed that an integral was already defined for some class T_0 of functions of a sufficiently simple type. With that definition somehow given, it was shown how to extend the class of integrable functions without assuming any properties of the elements, p, of which the integrands are functions. By the very attitude taken up in that analysis, the usual derivative is excluded since it is defined in terms of sequences of "intervals" of certain types enclosing an element.

W. L. Hart† has recently considered a Pseudo-derivative which gives more freedom in the study of inverse integrals but even this depends on sequences of intervals. We shall use, instead, an idea suggested by a further paper‡ on the general integral. There it was proved, that if $h(p) \geqq 0$ is summable with respect to $S(f)$ and limited then there exists a function $\lambda(p)$ which is everywhere equal to 1 or -1 and such that, for every $f(p)$ which is S-summable and which vanishes with $h(p)$,

$$S(f) = I(\lambda f),$$

where $I(f)$ is the modular integral associated with $S(f)$.

Definition 1.1. If $J(f)$ is a non-negative integral, $S(f)$ another general integral, and if we can find a function, $\delta(p)$, such that for every $f(p)$, of a certain wide class,

$$S(f) = J(\delta f),$$

then we shall call $\delta(p)$ the *quasiderivative* of S with respect to J.

This definition corresponds to the relation concerning the general Stieltjes integral that if $F(e)$ is an additive function of sets which is absolutely continuous, then it possesses nearly everywhere a derivative, DF, such that

* P. J. Daniell, Annals of Math., vol. 19 (1918), p. 279.

† W. L. Hart, Bull. Amer. Math. Soc., vol. 27 (1921), p. 202.

‡ P. J. Daniell, Annals of Math., vol. 21 (1920), p. 203.

$$F(e) = \int_e DF\, dm(e), \qquad \int f(x)\, dF(e) = \int DF\, f(x)\, dm(e),$$

for all functions, $f(x)$, which are summable with respect to $F(e)$.

To simplify the task of the reader we repeat some of the statements made in the second paper to which we have referred.

The outer measure of a set, e, of elements, p, is defined as the upper semi-integral of the function equal to 1 on e and to 0 elsewhere. A set of zero outer I-measure is said to be of zero measure (I). Two functions are nearly equal (I) if the outer integral of their modular difference is zero. Two nearly equal functions are equal nearly everywhere (I).

Our analysis is based on the following theorems of the paper referred to:

1.5. If $h \geq 0$ is of class T_0, there exists a function, k, summable (I) such that $0 \leq k \leq h$ and

$$S(k) = I_1(h),$$

where I_1 is the positive integral, $I = I_1 + I_2$, is the modular integral associated with the general integral S.

Furthermore, a function $\theta(p)$ can be found such that

$$0 \leq \theta \leq 1, \qquad k = \theta h.$$

Now, as is shown in the paper,

$$I_1(h-k) = I_2(k) = 0.$$

Hence

$$S(\theta h) = S(k) = I_1(h),$$
$$S[(1-\theta)h] = S(h) - S(k) = S(h) - I_1(h) = -I_2(h).$$

1.6. If f is summable and vanishes with h,

$$S(\theta f) = I_1(f),$$
$$S[(1-\theta)f] = S(f) - I_1(f) = -I_2(f).$$

2. We shall first consider the simplest case.

Definition 2.1. Two integrals S_1, S_2 are said to be *T-connected* if every function which is a member of class $T_0(S_1)$ is summable with respect to S_2, and every member of $T_0(S_2)$ is summable S_1.

This can only be possible when the integrands of S_1, S_2 are functions of elements of the same set P_0.

Definition 2.2. If $J(f)$ is a positive integral, $S(f)$ a general integral, which are T-connected and if a number K exists such that for every function f of $T_0(S)$,

$$|S(f)| \leq KJ(|f|),$$

then S is said to be *dominated* by J.

THEOREM 2.3. *If S is dominated by J, any function which is summable J is summable S.*

LEMMA 2.31. *If I is the modular integral associated with S, and if $f \geq 0$ is of class $T_0(S)$ then $I(f)$ is the upper bound of $|S(\varphi)|$ for all functions, φ, of class $T_0(S)$ such that*

$$|\varphi| \leq f.$$

For we know, in the first place, that if $|\varphi| \leq f$,

$$|S(\varphi)| \leq I(|\varphi|) \leq I(f).$$

In the second place, by definition, $I_1(f)$ is the upper bound of $S(\psi)$ for all functions ψ which are members of T_0 and such that $0 \leq \psi \leq f$. Hence, given any $e > 0$, we can choose ψ so that

$$\begin{aligned} S(\psi) &> I_1(f) - e, \\ I(f) &= 2I_1(f) - S(f) \\ &< 2S(\psi) + 2e - S(f) \\ &= S(2\psi - f) + 2e. \end{aligned}$$

But $2\psi - f$ satisfies the conditions for a function φ in the lemma so that $I(f)$ is the upper bound of $S(\varphi)$, and, by choosing $-\varphi$ instead of φ, it is also the upper bound of $-S(\varphi)$ and of $|S(\varphi)|$.

We can now consider the main theorem to be proved. From the lemma it follows that

$$I(f) \leq KJ(f).$$

This inequality holds, at present, for functions of class $T_0(S)$. From the methods of extension of the general integral it follows that if f is non-negative and summable J,

$$\bar{I}(f) \leq K J(f).$$

But if f is any function which is summable J, given any $\epsilon > 0$, we can find a member, g, of class $T_0(J)$ such that

$$J(|f-g|) < \epsilon/2K, \qquad \bar{I}(|f-g|) < \epsilon/2.$$

But S, J are T-connected so that g is summable S and we can find ψ of class $T_0(S)$ such that

$$I(|g-\psi|) < \epsilon/2.$$

Hence

$$\bar{I}(|f-\psi|) < \epsilon,$$

so that f is summable S, and the theorem is proved. It is also evident that if f is summable J, and non-negative

$$I(f) \leq K J(f).$$

We are now in a position to prove the theorem which is the backbone of the present paper.

THEOREM 2.4. *If I, J are two T-connected positive integrals, and if for every non-negative function, f, of class $T_0(I)$*

$$I(f) \leq J(f),$$

then a function $\delta(p)$ can be found such that

$$0 \leq \delta(p) \leq 1, \qquad I(f) = J(f\delta),$$

for every function f which is summable J and which vanishes with a particular function, h, depending only on J.

Let λ be a given number between 0 and 1. Then

$$I(f) - \lambda J(f)$$

is a general S-integral dominated by $J(f)$.

Choose $h(p) \geq 0$, once for all, a function summable J, and therefore summable $(I - \lambda J)$. For added generality, but not of logical necessity, choose it so as to vanish at as few "points" as possible. By 1.5 we can find a function $\theta(p)$ equal everywhere to 0 or 1 such that if $f(p)$ is summable J and vanishes with $h(p)$ so also is $\theta(p) f(p)$ and

$$I(\theta f) - \lambda J(\theta f) = I_1(f),$$

$$I[(1-\theta)f] - \lambda J[(1-\theta)f] = -I_2(f).$$

It has not yet been proved that θ is everywhere equal to 0 or 1. However it lies between 0 and 1 inclusive so that $\theta(1-\theta)$ is non-negative. Again I_1, I_2 are positive integrals, so that if f is non-negative and summable J, replacing it by $(1-\theta)f$ in the former, and by θf in the latter of the equations above, it is evident that

$$I[\theta(1-\theta)f] - \lambda J[\theta(1-\theta)f]$$

is both non-negative and non-positive. This expression is therefore zero and $\theta(1-\theta)h$ is nearly everywhere zero with respect to $I - \lambda J$ since f is any summable non-negative function which vanishes with h. Wherever it is not zero replace it by 0 by defining $\theta = 0$ at such "points". Then our statement is fully justified, for such a change will not alter the two equations above.

In what follows it will be tacitly implied that the integrands are summable J and vanish with h.

We now see that to every λ between 0 and 1 there corresponds a function $\theta(p)$ such that

$$I(\theta f) - \lambda J(\theta f) \geq 0, \qquad I[(1-\theta)f] - \lambda J[(1-\theta)f] \leq 0.$$

Divide the interval (0,1) into 2^n equal parts and let

$$\lambda(n, r) = r/2^n.$$

A. For a particular value of n let it be assumed possible to find a set of functions $\varphi(n, r)$ everywhere equal to 0 or 1 such that for all p,

$$\varphi(n, 1) + \varphi(n, 2) + \cdots + \varphi(n, 2^n) = 1,$$

and, for all functions $f > 0$,

$$\lambda(n, r-1).J[\varphi(n, r)f] < I[\varphi(n, r)f] < \lambda(n, r).J[\varphi(n, r)f] \quad r = 1, 2, \cdots 2^n.$$

Given n and r and taking

$$\lambda = (r - \tfrac{1}{2})\, 2^n = \lambda(n+1, 2r-1)$$

in our two inequalities we can choose $\theta(p)$, depending also on n and r, so that

B. $$I(\theta f) - \lambda . J(\theta f) \geq 0,$$

C. $$I[(1-\theta)f] - \lambda . J[(1-\theta)f] \leq 0,$$

for all non-negative functions, f.

When $s = 2r$ choose

$$\varphi(n+1, s) = \theta \varphi(n, r),$$

and when $s = 2r-1$ choose

$$\varphi(n+1, s) = (1-\theta)\ \varphi(n, r).$$

Then $\varphi(n+1, s)$ is everywhere equal to 0 or 1 and

$$\varphi(n+1,1) + \varphi(n+1,2) + \cdots + \varphi(n+1, 2^{n+1}) = 1$$

everywhere.

From B, putting $\varphi(n, r)f$ instead of f, when $s = 2r$,

$$I[\varphi(n+1, s)f] > \lambda(n+1, s-1)\ J[\varphi(n+1, s)f],$$

and from A putting θf instead of f,

$$I[\varphi(n+1, s)f] < \lambda(n+1, s)\ .J[\varphi(n+1, s)f].$$

Again when $s = 2r-1$, putting $\varphi(n, r)f$ for f in C,

$$I[\varphi(n+1, s)f] \leq \lambda(n+1, s)\ I[\varphi(n+1, s)f],$$

and in A using $(1-\theta)f$ instead of f,

$$\lambda(n+1, s-1)\ I[\varphi(n+1, s)f] \leq I[\varphi(n+1, s)f].$$

This shows that if A is satisfied for any n it is also true for $n+1$. But when $n = 0$ we can choose $\varphi(0, 1) = 1$ for all p and then

$$0\ I[\varphi(0, 1)f] \leq I[\varphi(0, 1)f] \leq I[\varphi(0, 1)f],$$

so that A is certainly satisfied for $n = 0$ and therefore for all values of n (integral, of course).

Let

$$\psi(n) = \sum_r \lambda(n, r-1)\,\varphi(n, r), \qquad r = 1 \text{ to } 2^n,$$

$$\sigma(n) = \sum_r \lambda(n, r)\,\varphi(n, r), \qquad r = 1 \text{ to } 2^n.$$

At a subdivision when $\psi(n)$ is replaced by $\psi(n+1)$ the term

$$(r-1)/2^n\,\varphi(n, r)$$

is replaced by the sum of two terms,

$$(2r-1)/2^{n+1}\,\theta\,\varphi(n, r) + (2r-2)/2^{n+1}(1-\theta)\varphi(n, r)$$

$$= (r-1)/2^n\varphi(n, r)\,[1 + \theta/2(r-1)].$$

Hence $\psi(n) \leq \psi(n+1)$, and, as $\psi(n) \leq 1$,

$$\lim \psi(n) = \delta$$

exists for each "point" p, and

$$0 \leq \delta(p) \leq 1, \qquad \psi(n) \leq \delta.$$

Similarly $\sigma(n) \geq \sigma(n+1)$, and as

$$\sigma(n) - \psi(m) = 1/2^n, \qquad \lim \sigma(n) = \delta, \ \sigma(n) \geq \delta.$$

Adding up the inequalities of A for $r = 1$ to 2^n,

$$\sum_r \lambda(n, r-1)\, J[q(n, r)f] \leq \sum_r I[q(n, r)f] \leq \sum_r \lambda(n, r)\, J[q(n, r)f],$$

that is

$$J[\psi(n)f] \leq I(f) \leq J[\sigma(n)f]$$

for all non-negative f. It follows from the nature of the general integral that $\lim \psi(n) = \lim \sigma(n) = \delta$ is summable J and

$$J(\delta f) \leq I(f) \leq J(\delta f),$$

which means that $J(\delta f) = I(f)$.

Since any function is the difference of two non-negative functions, which are summable if the original function is summable, if $f(p)$ is summable J and vanishes with h,

$$I(f) = J(\delta f).$$

δf is summable and $0 \leq \delta \leq 1$. This proves the theorem.

THEOREM 2.5. *If a general integral $S(f)$ is dominated by the positive integral $J(f)$, given some function $h(p) \geq 0$, which is summable J, a function $\delta(p)$ can be found such that for all functions, f, which are summable J and vanish with h, $\delta(p)\, f(p)$ is summable J and*

$$S(f) = J(\delta f).$$

Let I_1, I_2 denote the positive integrals associated with S and, if

$$|S(f)| \leq KJ(f), \qquad f \geq 0,$$

define

$$I_3(f) = (1/K)\, I_1(f), \qquad I_4(f) = (1/K)\, I_2(f).$$

Then I_3, I_4 are related to J as I is to J in the previous theorem and

$$I_3(f) = J(\delta_3 f), \qquad I_4(f) = J(\delta_4 f).$$

If

$$\delta = K[\delta_3 - \delta_4],$$

$$S(f) = K[I_3(f) - I_4(f)] = J(\delta f).$$

Since δ_3, δ_4 are non-negative and do not exceed 1,

$$|\delta(p)| \leqq K.$$

3. We pass now to cases corresponding to *absolute continuity* in the theory of the Lebesgue integral.

Definition 3.1. A positive integral $I(f)$ is said to be *based on* $J(f)$ if they are T-connected and if for any non-negative function f such that $J(f) = 0$,

$$I(f) = 0.$$

Definition 3.2. A general integral $S(f)$ is said to be based on $J(f)$ if its modular integral is based on $J(f)$.

If the integral $I(f)$ is based on $J(f)$ the two integrals are T-connected and there are certainly functions which are summable with respect to both I and J, and, therefore, with respect to $I+J$. Let $h(p) \geq 0$ be such a function.

THEOREM 3.3. *If the positive integral I is based on J, then a function $\delta(p)$ can be found such that, for all functions, f, summable $(I+J)$ which vanish with $h(p)$*

$$I(f) = J(f\delta),$$

where $h(p)$ is a given non-negative function which is summable with respect to $I+J$, independent of $f(p)$.

In the first place, $I - \lambda J$ is dominated by $I+J$ for all values of λ. For each integral value of n we can choose the function $\theta(n)$ of the variable element p so that

$$I[\theta(n)f] \geq n\,J[\theta(n)f], \qquad I[f-\theta(n)f] \leqq n\,J[f-\theta(n)f].$$

provided f is non-negative and summable $(I+J)$ and vanishes with h.

Let

$$\varphi(n) = \theta(1)\,\theta(2)\cdots\theta(n).$$

Then

$$\varphi(n) - \varphi(n+1) = \varphi(n-1)\,\theta(n)\,[1-\theta(n+1)] \geq 0.$$

In the earlier of the inequalities above replace f by $f\varphi(n-1)[1-\theta(n+1)]$. Then, if

$$J_n(f) = J[f\varphi(n)-f\varphi(n+1)],$$

$$I_n(f) = I[f\varphi(n)-f\varphi(n+1)]-nJ[f\varphi(n)-f\varphi(n+1)].$$

$J_n(f)$ and $I_n(f)$ are positive integrals. Let their fundamental classes of integrands, $T_0(I_n)$, $T_0(J_n)$, be chosen as that of J, namely $T_0(J)$. Then the two integrals are T-connected.

Again, in the second inequality replace n by $n+1$ and f by $f\varphi(n)$. Then

$$I_n(f) \leq J_n(f)$$

if f is non-negative, so that I_n, J_n satisfy the conditions of Theorem 2.4. Hence a function $\delta(n)$ can be found such that it lies between 0 and 1 and

$$I_n(f) = J_n[\delta(n)f].$$

In other symbols,

$$I[f\varphi(n)-f\varphi(n+1)] = J[f(n+\delta[n])(\varphi[n]-\varphi[n+1])].$$

Let

$$\gamma_n = \delta(0)[1-\varphi(1)]+[1+\delta(1)][\varphi(1)-\varphi(2)]+\cdots$$
$$+[n-1+\delta(n-1)][\varphi(n-1)-\varphi(n)].$$

Then, by addition of the previous equations for $n = 0$ to $n-1$,

$$I[f-f\varphi(n)] = J(\gamma_n f).$$

Now the sequence $\varphi(n)$ is non-increasing so that if f is non-negative and summable $(I+J)$, by a fundamental property of the general integral, if $\lim \varphi(n) = \varphi$,

$$I[(1-\varphi)f] = \lim I[f-f\varphi(n)]$$

exists, so that $J(\gamma_n f)$ converges. Since, again, the sequence γ_n is non-decreasing, if $\delta(p) = \lim \gamma_n(p)$,

$$I[(1-\varphi)f] = J(\delta f).$$

Wherever $\delta(\rho)$ is infinite we can, if we choose, replace it by 0 without affecting the integral, since δf is summable with respect to J. Now

$$I[\varphi(n)f] \geq n J[\varphi(n)f],$$

since $\varphi(n)f = \theta(n) \times \varphi(n-1)f$. Therefore

$$J[\varphi(n)f] \leq I[\varphi(n)f]/n \leq I(f)/n.$$

Therefore, in the limit,

$$J(\varphi f) = 0.$$

So far we have only considered non-negative functions f. But any summable function is the difference of two non-negative summable functions, so that we obtain a theorem slightly more general than that stated at the beginning of this section.

If I und J are two positive T-connected integrals, and if h is a non-negative function summable $(I+J)$ than a function δ can be found such that for any function f which vanishes with h and which is summable $(I+J)$,

$$I[(1-\varphi)f] = J(f\delta),$$

where $f\delta$ is summable J and where φ is a function everywhere equal to 0 or 1, independent of f such that

$$J(\varphi f) = 0.$$

Evidently φ vanishes everywhere except perhaps where h vanishes or at elements of a set whose measure with respect to J is zero.

Thus far the condition that I be based on J has not been used except in the use of the fact that they are T-connected. But if I is based on J, it follows that

$$I(\varphi f) = 0, \qquad I(f) = I[f-\varphi f] = J(f\delta),$$

and the theorem is proved.

Now, from the definitions, if an integral $S(f)$ is dominated by a positive integral $J(f)$ and if $J(f)$ is based on $K(f)$, then $S(f)$ is based on $K(f)$.

But if $S(f)$ is a general integral its two associated positive integrals, I_1, I_2 are dominated by its modular integral $I(f)$, and if the latter is based on $J(f)$ so also are $I_1(f)$, $I_2(f)$. We thus obtain an immediate corollary of the preceding theorem.

THEOREM 3.4. *If the general integral $S(f)$ is based on $J(f)$, and if $h(p)$ is a non-negative function which is summable $(I+J)$ where $I(f)$ is the modular integral associated with $S(f)$, then a function $\delta(p)$ can be found such that for all functions f, which vanish with h and are summable $(I+J)$,*

$$I(f) = J(\delta f).$$

Such a function $\delta(p)$ is, then, a quasi-derivative of S with respect to J.

This type of derivative can be used even in cases of integrals in a space of a countable infinity of dimensions or in a space of which the elements p are continuous functions of a variable t.

RICE INSTITUTE
HOUSTON, TEXAS.

ON A CLASS OF TRANSFORMATIONS IN FUNCTION SPACE.

By I. A. Barnett.

Let $\varphi(x)$ be a continuous function on the range I: $0 \leqq x \leqq 1$, and let $\mathfrak{P}(x, \varphi)$ be a Schmidt integral-power-series.* The transformation

$$\delta\varphi(x) = \mathfrak{P}(x, \varphi)\,\delta t \tag{1}$$

where δt is an infinitesimal, has been called by Kowalewski,† a *regular infinitesimal transformation*. The purpose of this paper is to find the form of $\mathfrak{P}$ in order that the transformation (1) take every infinitesimal triangle in function space of a given area into a triangle having the same area.

In § 1, a necessary condition is found in order that (1) be an area preserving transformation in the sense described. In § 2 the explicit form of $\mathfrak{P}$ is investigated and it is shown (§ 3) that these transformations form a group in the sense defined by Kowalewski.

In § 4 the group of motion of function space is defined as all the regular infinitesimal transformations which leave invariant

$$\int_0^1 [\varphi(x) - \psi(x)]^2\,dx$$

where φ and ψ are arbitrary continuous functions, and it is seen that this group is the same as the area preserving group. It is proved that the radii of curvature of a curve in function space (Kowalewski) are invariant under this group.

1. **A necessary condition.** Analogous to the situation in n-space, a triangle in function space, i. e., in the space of all continuous functions having the range I, is defined by

$$\varphi(x), \qquad \varphi(x) + \alpha(x), \qquad \varphi(x) + \beta(x) \tag{2}$$

and its area is given by

$$\frac{1}{2}\sqrt{\int_0^1 \alpha^2(x)\,dx \int_0^1 \beta^2(x)\,dx - \left(\int_0^1 \alpha(x)\,\beta(x)\,dx\right)^2}\,. \tag{3}$$

* See Erhard Schmidt, Zur Theorie der linearen und nichtlinearen Integralgleichungen, III. Teil, Math. Annalen, vol. 65, p. 370. Also, G. Kowalewski, Über Funktionräume (II. Mitteilung), Wien. Ber., vol. 120, p. 1435.

† Loc. cit. § 3.

By means of the transformation (1), the triangle (2) goes into another triangle

$$\varphi_1(x), \qquad \varphi_1(x)+\alpha_1(x), \qquad \varphi_1(x)+\beta_1(x),$$

where

$$(4)\qquad \begin{cases} \varphi_1(x) = \varphi(x)+\mathfrak{P}(x,\varphi)\,\delta t, \\ \varphi_1(x)+\alpha_1(x) = \varphi(x)+\alpha(x)+\mathfrak{P}(x,\varphi+\alpha)\,\delta t, \\ \varphi_1(x)+\beta_1(x) = \varphi(x)+\beta(x)+\mathfrak{P}(x,\varphi+\beta)\,\delta t. \end{cases}$$

Now Kowalewski has shown* that there exists an expansion

$$\mathfrak{P}(x,\varphi+\gamma) = \mathfrak{P}(x,\varphi)+\frac{\mathfrak{P}(x,\varphi;\gamma)}{1!}+\frac{\mathfrak{P}(x,\varphi;\gamma,\gamma)}{2!}+\cdots$$

provided $|\varphi|$, $|\gamma|$ are sufficiently small for all x's of the range I. Here, $\mathfrak{P}(x,\varphi;\gamma)$, $\mathfrak{P}(x,\varphi;\gamma,\gamma)$, ... are the first, second, ... differentials of $\mathfrak{P}(x,\varphi)$ for the argument function γ, and may be computed by means of the formulas

$$\mathfrak{P}(x,\varphi;\gamma) = \frac{d}{du}\mathfrak{P}(x,\varphi+u\gamma)_{u=0}$$

$$\mathfrak{P}(x,\varphi;\gamma,\gamma) = \frac{d}{du}\mathfrak{P}(x,\varphi+u\gamma;\gamma)_{u=0}.$$

It is readily seen that $\mathfrak{P}(x,\varphi;\gamma)$ is a linear functional of γ, $\mathfrak{P}(x,\varphi;\gamma,\gamma)$ a quadratic functional of γ, etc. Equations (4) may now be written

$$\begin{aligned} \varphi_1(x) &= \varphi(x)+\mathfrak{P}(x,\varphi)\,\delta t, \\ \varphi_1(x)+\alpha_1(x) &= \varphi(x)+\alpha(x)+[\mathfrak{P}(x,\varphi)+\mathfrak{P}(x,\varphi;\alpha)]\,\delta t, \\ \varphi_1(x)+\beta_1(x) &= \varphi(x)+\beta(x)+[\mathfrak{P}(x,\varphi)+\mathfrak{P}(x,\varphi;\beta)]\,\delta t, \end{aligned}$$

if we assume that $|\alpha|$, $|\beta|$, $|\varphi|$ are sufficiently small so that the differentials higher than the first may be disregarded since they are at least quadratic functionals of α and β. The preceding equations reduce to

$$(5)\qquad \begin{cases} \alpha_1(x) = \alpha(x)+\mathfrak{P}(x,\varphi;\alpha)\,\delta t, \\ \beta_1(x) = \beta(x)+\mathfrak{P}(x,\varphi;\beta)\,\delta t. \end{cases}$$

* Loc. cit. formula 6".

It is required that

$$\int_0^1 \alpha_1^2(x)\,dx \int_0^1 \beta_1^2(x)\,dx - \left(\int \alpha_1(x)\beta_1(x)\,dx\right)^2 = \int_0^1 \alpha^2(x)\,dx \int_0^1 \beta^2(x)\,dx - \left(\int_0^1 \alpha(x)\beta(x)\,dx\right)^2$$

shall be an identity for all φ, α, β whose absolute values are sufficiently small, where it is understood that the in the left hand member, α_1 and β_1 are replaced by (5). Expanding this out, and equating powers of δt, one must have as a necessary condition the identity

$$\int_0^1 \alpha^2(x)\,dx \int_0^1 \beta(x)\,\mathfrak{P}(x,\varphi;\beta)\,dx + \int_0^1 \beta^2(x)\,dx \int_0^1 \alpha(x)\,\mathfrak{P}(x,\varphi;\alpha)\,dx - \int_0^1 \alpha(x)\beta(x)\,dx\left[\int_0^1 \alpha(x)\,\mathfrak{P}(x,\varphi;\beta)\,dx + \int_0^1 \beta(x)\,\mathfrak{P}(x,\varphi;\alpha)\,dx\right] = 0. \tag{6}$$

THEOREM. *A necessary condition that a regular infinitesimal transformation* (1) *take every infinitesimal triangle of function space into one of the same area is that* (6) *be an identity for all continuous* α, β *of sufficiently small moduli.*

Let $\lambda(x)$ and $\mu(x)$ be a particular choice of $\alpha(x)$ and $\beta(x)$ respectively, satisfying the following two conditions:

$$1)\quad \int_0^1 \lambda(x)\mu(x)\,dx = 0,$$

$$2)\quad \int_0^1 \lambda^2(x)\,dx = \int_0^1 \mu^2(x)\,dx.$$

Then (6) becomes

$$\int_0^1 \lambda(x)\,P(x,\varphi;\lambda)\,dx + \int_0^1 \mu(x)\,P(x,\varphi;\mu)\,dx = 0. \tag{7}$$

The problem is now reduced to finding the form of the integral-power-series $\mathfrak{P}(x,\varphi)$ which satisfies the identity (7) for all continuous functions λ and μ of sufficiently small moduli which are orthogonal and have equal norms.

2. **The explicit form of the integral-power-series $\mathfrak{P}(x,\varphi)$.** It is desired in this section to find the explicit expresssion for $\mathfrak{P}(x,\varphi)$ in order

that the infinitesimal transformation (1) preserve areas in the sense described in the preceding section. Every Schmidt integral power series has the form

$$\mathfrak{P}(x,\varphi) = \mathfrak{A}_0(x,\varphi) + \mathfrak{A}_1(x,\varphi) + \mathfrak{A}_2(x,\varphi) + \cdots \tag{8}$$

where

$$\begin{cases} \mathfrak{A}_0(x,\varphi) = \alpha(x), \\ \mathfrak{A}_1(x,\varphi) = \beta(x)\varphi(x) + \int_0^1 \gamma(x,y)\varphi(y)\,dy, \\ \mathfrak{A}_2(x,\varphi) = \varepsilon(x)\varphi^2(x) + \varphi(x)\int_0^1 \theta(x,y)\varphi(y)\,dy \\ \qquad\qquad + \int_0^1 \zeta(x,y)\varphi^2(y)\,dy + \int_0^1\int_0^1 \chi(x,y,z)\varphi(y)\varphi(z)\,dy\,dz, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases} \tag{9}$$

Here, $\alpha, \beta, \gamma, \ldots$ are all arbitrary continuous functions of their arguments, all of which have the range I. The problem is to determine $\alpha, \beta, \gamma, \ldots$ in order that (7) be satisfied. In view of (8) the condition (7) may be written

$$\int_0^1 \lambda(x)\left[\mathfrak{A}_1(x,\varphi;\lambda) + \mathfrak{A}_2(x,\varphi;\lambda) + \cdots\right]dx + \int_0^1 \mu(x)\left[\mathfrak{A}_1(x,\varphi;\mu) + \mathfrak{A}_2(x,\varphi;\mu) + \cdots\right]dx = 0.$$

If, in this identity, one replaces φ by $m\varphi$, where m is an arbitrary real number, it must also be true. Since $\mathfrak{A}_1(x, m\varphi;\lambda)$ contains m to the first power, $\mathfrak{A}_2(x, m\varphi;\lambda)$ contains m to the second power, etc., it follows that the preceding identity is equivalent to the following identities

$$\int_0^1 \lambda(x)\,\mathfrak{A}_p(x,\varphi;\lambda)\,dx + \int_0^1 \mu(x)\,\mathfrak{A}_p(x,\varphi;\mu)\,dx = 0, \quad p = 1, 2, 3\cdots. \tag{10}$$

Consider, first, the integral power form of the first order

$$\mathfrak{A}_1(x,\varphi) = \alpha(x)\varphi(x) + \int_0^1 \beta(x,y)\varphi(y)\,dy.$$

The condition (10) for $p = 1$, yields

$$\int_0^1 [\lambda^2(x)+\mu^2(x)]\alpha(x)\,dx + \int_0^1\int_0^1 \beta(x,y)[\lambda(x)\lambda(y)+\mu(x)\mu(y)]\,dx\,dy = 0. \tag{11}$$

This is to be an identity for all λ, μ satisfying $\int_0^1 \lambda(x)\,\mu(x)\,dx = 0$ and $\int_0^1 \lambda^2(x)\,dx = \int_0^1 \mu^2(x)\,dx$ and it is desired to determine $\alpha(x)$ and $\beta(x, y)$. Let x_1, y_1 be two distinct values of x on the range I, and let a positive number δ be chosen so that the intervals $(x_1 - \delta, x_1 + \delta)$, $(y_1 - \delta, y_1 + \delta)$ do not overlap. Now define the continuous functions $\lambda(x)$, $\mu(x)$ in the following way,

$$\lambda(x) = \begin{cases} 0, & 0 \leqq x < x_1 - \delta, \quad x_1 + \delta < x \leqq 1, \\ \varepsilon \cos \dfrac{\pi}{2\delta}(x - x_1), & x_1 - \delta \leqq x \leqq x_1 + \delta, \end{cases}$$

$$\mu(x) = \begin{cases} 0, & 0 \leqq x < y_1 - \delta, \quad y_1 + \delta < x \leqq 1, \\ \varepsilon \cos \dfrac{\pi}{2\delta}(x - y_1), & y_1 - \delta \leqq x \leqq y_1 + \delta, \end{cases}$$

where ε is a sufficiently small positive number. It is clear that all the conditions on λ and μ are fulfilled. Equation (11) may now be written

$$\int_{x_1-\delta}^{x_1+\delta} \alpha(x) \cos^2 \frac{\pi}{2\delta}(x - x_1)\,dx + \int_{y_1-\delta}^{y_1+\delta} \alpha(x) \cos^2 \frac{\pi}{2\delta}(x - y_1)\,dx$$

$$+ \int_{x_1-\delta}^{x_1+\delta} dx \int_{y_1-\delta}^{y_1+\delta} dy\, \beta(x y) \left[\cos \frac{\pi}{2\delta}(x - x_1) \cos \frac{\pi}{2\delta}(y - x_1) \right.$$

$$\left. + \cos \frac{\pi}{2\delta}(x - y_1) \cos \frac{\pi}{2\delta}(y - y_1) \right] = 0.$$

By the Mean Value Theorem of definite integrals this may be written

$$\alpha(\xi) \int_{x_1-\delta}^{x_1+\delta} \cos^2 \frac{\pi}{2\delta}(x - x_1)\,dx + \alpha(\eta) \int_{y_1-\delta}^{y_1+\delta} \cos^2 \frac{\pi}{2\delta}(x - y_1)\,dx$$

$$+ \beta(\xi', \eta') \int_{x_1-\delta}^{x_1+\delta} dx \int_{y_1-\delta}^{y_1+\delta} dy \left[\cos \frac{\pi}{2\delta}(x - x_1) \cos \frac{\pi}{2\delta}(y - x_1) \right.$$

$$\left. + \cos \frac{\pi}{2\delta}(x - y) \cos \frac{\pi}{2\delta}(y - y_1) \right] = 0,$$

since the functions under the integrals do not change sign in the intervals under consideration. Here (ξ, ξ') and (η, η') are suitably chosen values of x

in $(x_1-\delta,\ x_1+\delta)$, $(y_1-\delta,\ y_1+\delta)$ respectively. When the integrations are carried out, the preceding may be written

$$2\delta\alpha(\xi)+2\delta\alpha(\eta)+\frac{32\delta^2}{\pi^2}\cos\frac{\pi}{2\delta}(y_1-x_1)=0.$$

Dividing by δ and letting $\delta\to 0$, one obtains, since $\alpha(x)$ is a continuous function, $\alpha(x_1)+\alpha(y_1)=0$ for all $x_1\neq y_1$ of $0\leq x\leq 1$, and hence $\alpha(x)\equiv 0$. Equation (11) now reduces to

$$\int_0^1\int_0^1[\lambda(x)\,\lambda(y)+\mu(x)\,\mu(y)]\,\beta(xy)\,dx\,dy=0$$

which may also be written

$$\int_0^1\int_0^1\tfrac{1}{2}(\beta(xy)+\beta(yx))\,[\lambda(x)\,\lambda(y)+\mu(x)\,\mu(y)]\,dx\,dy=0.$$

Proceeding as above one readily finds

$$\beta(x_1,y_1)+\beta(y_1,x_1)=0, \qquad (x_1\neq y_1), \tag{12}$$

and, because of the continuity of the function β, it follows that $\beta(x_1,x_1)+\beta(x_1,x_1)=0$.

Hence, one has the following result.

The most general integral power form of the first order which satisfies condition (7) *has the form*

$$\int_0^1\beta(x,y)\,\varphi(y)\,dy \tag{13}$$

where β *is a skew-symmetric function, i. e.,*

$$\beta(x,y)+\beta(y,x)=0 \qquad 0\leq x\leq 1,\ 0\leq y\leq 1.$$

To investigate the integral power forms of orders higher than the first, one makes use of a device employed by Kowalewski* in connection with the projective and conformal groups in function space. If in the identities (10), one replaces φ by $\varphi+m\psi$, where ψ is an arbitrary con-

* Loc. cit., §§ 7, 9.

tinuous function, and compares the coefficients of equal powers of m, one obtains

$$\int_0^1 \lambda(x)\, \mathfrak{A}_p(x, \varphi; \lambda, \psi)\, dx + \int_0^1 \mu(x)\, \mathfrak{A}_p(x, \varphi; \mu, \psi)\, dx = 0,$$

where $\mathfrak{A}_p(x, \varphi; \lambda, \psi)$ is the first differential of $\mathfrak{A}_p(x, \varphi; \lambda)$ with the argument function ψ. Since by definition

$$\mathfrak{A}_p(x, \varphi; \lambda, \psi) = \frac{d}{du}\frac{d}{dv}\mathfrak{A}_p(x, \varphi + \lambda u + \psi v)\Big|_{\substack{u=0\\v=0}},$$

it follows that

$$\mathfrak{A}_p(x, \varphi; \lambda, \psi) = \mathfrak{A}_p(x, \varphi; \psi, \lambda).$$

Hence, from (10) one obtains

$$\int_0^1 \lambda(x)\, \mathfrak{A}_p(x, \varphi; \psi, \lambda)\, dx + \int_0^1 \mu(x)\, \mathfrak{A}_p(x, \varphi; \psi, \mu)\, dx = 0 \tag{14}$$

or

$$\int_0^1 \lambda(x)\, \mathfrak{B}_{p-1}(x, \varphi; \lambda)\, dx + \int_0^1 \mu(x)\, \mathfrak{B}_{p-1}(x, \varphi; \mu)\, dx = 0$$

where

$$\mathfrak{B}_{p-1}(x, \varphi) = \mathfrak{A}_p(x, \varphi; \psi).$$

In particular if $p = 2$, one must have

$$\int_0^1 \lambda(x)\, \mathfrak{B}_1(x, \varphi; \lambda)\, dx + \int_0^1 \mu(x)\, \mathfrak{B}_1(x, \varphi; \mu)\, dx = 0$$

i. e., $\mathfrak{B}_1(x, \varphi)$ must satisfy the first identity of (10) for every continuous function ψ and hence must be of the form (13). In other words, $\mathfrak{A}_2(x, \varphi; \psi)$ must have the form (13) for every continuous function $\psi(x)$. But,

$$\mathfrak{A}_2(x, \varphi) = \gamma(x)\, \varphi^2(x) + \varphi(x) \int_0^1 \varepsilon(x, y)\, \varphi(y)\, dy$$

$$+ \int_0^1 \theta(x, y)\, \varphi^2(y)\, dy + \int_0^1 \int_0^1 \zeta(x, y, z)\, \varphi(y)\, \varphi(z)\, dy\, dz$$

where one may suppose $\zeta(xyz) = \zeta(xzy)$, which is obviously no restriction on the form of $\mathfrak{A}_2(x, \varphi)$. Hence,

$$\mathfrak{A}_2(x, \varphi; \psi) = 2\gamma(x)\,\psi(x)\,\varphi(x) + \varphi(x)\int_0^1 \varepsilon(x, y)\,\psi(y)\,dy$$

$$+ \psi(x)\int_0^1 \varepsilon(x, y)\,\varphi(y)\,dy + 2\int_0^1 \theta(x, y)\,\varphi(y)\,\psi(y)\,dy \tag{15}$$

$$+ 2\int_0^1\int_0^1 \zeta(x, y, z)\,\varphi(y)\,\psi(z)\,dy\,dz = \int_0^1 \beta(x, y)\,\varphi(y)\,dy$$

for every continuous function ψ. Now, consider a particular $\psi(x)$ defined as follows:

$\psi(x) = l$ (arbitrary positive number) $x_1 - \delta \leq x \leq x_1 + \delta$, and outside of this interval $\psi(x)$ lies between definite bounds independent of δ, but always remaining continuous. Then, if one allow $\delta \to 0$, (15) becomes

$$2l\gamma(x)\,\varphi(x) + \varphi(x)\int_0^1 \varepsilon(x, y)\,\psi(y)\,dy + l\int_0^1 \varepsilon(x, y)\,\varphi(y)\,dy$$

$$+ 2\int_0^1 \theta(x, y)\,\varphi(y)\,\psi(y)\,dy + 2\int_0^1\int_0^1 \zeta(x, y, z)\,\varphi(y)\,\psi(z)\,dy\,dz \tag{16}$$

$$= \int_0^1 \beta(x, y)\,\varphi(y)\,dy.$$

Since l is arbitrary, it follows in the first place that

$$2\gamma(x)\,\varphi(x) + \int_0^1 \varepsilon(x, y)\,\varphi(y)\,dy = 0$$

must be true for every continuous function φ with sufficiently small modulus. By choosing φ properly, as was done in the case of ψ above, one gets immediately that

$$\gamma(x) = 0, \qquad \varepsilon(x, y) = 0.$$

Hence, (16) becomes

$$\int_0^1 \varphi(y)\,dy\left[2\theta(x, y)\,\psi(y) - \beta(x, y) + 2\int_0^1 \zeta(x, y, z)\,\psi(z)\,dz\right] = 0$$

and since this is to be an identity in φ, it follows that

$$\beta(x, y) - 2\theta(x, y)\psi(y) - 2\int_0^1 \zeta(x, y, z)\psi(z)\, dz = 0$$

must be an identity in ψ, and by an argument similar to the above it follows that $\theta(x, y) \equiv 0$. Hence

$$\beta(x, y) = 2\int_0^1 \zeta(x, y, z)\psi(z)\, dz$$

for every continuous function ψ. But since $\beta(x, y)$ is skew-symmetric, it follows that $\zeta(x, y, z) \equiv -\zeta(y, x, z)$. It would follow therefore, since $\zeta(x, y, z) \equiv \zeta(x, z, y)$, that

$$\begin{aligned}\zeta(x, y, z) &= -\zeta(y, x, z) = -\zeta(y, z, x) = \zeta(z, y, x)\\ &= \zeta(z, x, y) = -\zeta(x, z, y) = -\zeta(x, y, z)\end{aligned}$$

and hence $\zeta(x, y, z) \equiv 0$.

If now one applies to the identities (14) the same argument that was used for equations (10), one sees that $\mathfrak{A}_3(x, \varphi; \psi, \chi)$ must have the same form as $\mathfrak{A}_2(x, \varphi)$, viz., $\mathfrak{A}_3(x, \varphi; \psi, \chi) \equiv 0$ for all φ, ψ, χ. In particular, if $\varphi = \psi = \chi$, $\mathfrak{A}_3(x, \varphi, \varphi, \varphi) \equiv 0$ for all φ. But if one replaces φ in $\mathfrak{A}_3(x, \varphi, \varphi, \varphi)$ by $m\varphi$, it is clear that

$$\mathfrak{A}_3(x, \varphi, \varphi, \varphi) = 3 \cdot 2\,\mathfrak{A}_3(x, \varphi)$$

and hence $\mathfrak{A}_3(x, \varphi) \equiv 0$. In a similar fashion all integral-power-forms of higher order can be proved identically zero.

Since an integral-power-form of the zero'th order, is not at all restricted by condition (7) one has proved the following

THEOREM. *Every regular infinitesimal transformation which satisfies* (7) *must have the form*

$$\alpha(x) + \int_0^1 \beta(x, y)\varphi(y)\, dy \tag{17}$$

where $\beta(x, y) = -\beta(y, x)$ *and* $\alpha(x)$ *is an arbitrary continuous function.*

3. **The sufficiency of the condition.** It will now be shown that if an infinitesimal transformation has the form

$$\delta\varphi = \left[\alpha(x) + \int_0^1 \beta(x, y)\varphi(y)\, dy\right]\delta t.$$

where $\beta(x, y) = -\beta(y, x)$, this transformation will actually have the property of transforming all infinitesimal triangles into triangles of equal area. For consider

$$A = \frac{1}{2}\sqrt{\int_0^1 \varrho^2(x)\,dx \int_0^1 \sigma^2(x)\,dx - \left(\int \varrho(x)\,\sigma(x)\,dx\right)^2}$$

where ϱ, σ are any continuous functions on I. Hence

$$\begin{aligned} 4A\,\delta A = &\int_0^1 \varrho^2(x)\,dx \int_0^1 \sigma(x)\,\delta\sigma(x)\,dx + \int_0^1 \sigma^2(x)\,dx \int_0^1 \varrho(x)\,\delta\varrho(x)\,dx \\ &- \int_0^1 \sigma(x)\,\varrho(x)\,dx\left(\int_0^1 \varrho(x)\,\delta\sigma(x)\,dx + \int_0^1 \sigma(x)\,\delta\varrho(x)\,dx\right). \end{aligned} \tag{18}$$

But, since

$$\mathfrak{P}(x, q) = \alpha(x) + \int \beta(x, y)\,q(y)\,dy,$$

it follows by (5) that

$$\delta\varrho(x) = \int_0^1 \beta(x, y)\,\varrho(y)\,dy, \quad \delta\sigma(x) = \int_0^1 \beta(x, y)\,\sigma(y)\,dy.$$

Substituting these expressions in (18), one gets

$$\begin{aligned} 4A\,\delta\Delta = &\int_0^1 \varrho^2(x)\,dx \int_0^1 \int_0^1 \beta(x, y)\,\sigma(x)\,\sigma(y)\,dx\,dy \\ &+ \int_0^1 \sigma^2(x)\,dx \int_0^1 \int_0^1 \beta(x, y)\,\varrho(x)\,\varrho(y)\,dx\,dy \\ &- \int_0^1 \varrho(x)\,\sigma(x)\,dx\left[\int_0^1 \int_0^1 \beta(x, y)\,\{\sigma(x)\,\varrho(y) + \varrho(x)\,\sigma(y)\}\,dx\,dy\right]. \end{aligned} \tag{19}$$

Now it is clear that if $\theta(x, y)$ is any symmetric continuous function of x, y then

$$\int_0^1 \int_0^1 \beta(x, y)\,\theta(x, y)\,dx\,dy = 0;$$

for

$$\int_0^1 \int_0^1 \beta(x, y)\, \theta(x, y)\, dx\, dy$$

$$= \frac{1}{2} \int_0^1 \int_0^1 \beta(x, y)\, \theta(xy)\, dx\, dy + \frac{1}{2} \int_0^1 \int_0^1 \beta(y, x)\, \theta(x, y)\, dx\, dy$$

$$- \frac{1}{2} \int_0^1 \int_0^1 \{\beta(x, y) + \beta(y, x)\}\, \theta(x, y)\, dx\, dy = 0$$

since $\beta(x, y) + \beta(y, x) = 0$. Therefore the right hand member of (19) vanishes and $\delta A = 0$ as desired.

4. **The group of area-preserving transformations in function space.** Let

$$\mathfrak{P}_1(x, \varphi) + \mathfrak{P}_2(x, \varphi) + \mathfrak{P}_3(x, \varphi) + \cdots \tag{20}$$

be a sum of regular integral-power-series. If

$$(\mathfrak{P}_i; \mathfrak{P}_j) = \mathfrak{P}_i(x, \varphi; \mathfrak{P}_j) - \mathfrak{P}_j(x, \varphi; \mathfrak{P}_i)$$

is of the form (20) for every i, j, then the infinitesimal transformations

$$\delta \varphi = (\mathfrak{P}_1 + \mathfrak{P}_2 + \mathfrak{P}_3 + \cdots)\, \delta t$$

are said to form a group.* The notation $\mathfrak{P}(x, \varphi; Q)$ means

$$\frac{d}{du} \mathfrak{P}[x, \varphi + u Q(x, \varphi)]_{u=0}.$$

It is seen immediately that

$$\delta \varphi = \left[\alpha(x) + \int_0^1 \beta(x, y)\, \varphi(y)\, dy\right] \delta t$$

* Kowalewski, loc. cit., § 6.

form a group. For

$$(\alpha_1; \alpha_2) = 0,$$

$$\left(\alpha; \int_0^1 \beta(x, y)\, \eta(y)\, dy\right) = \int_0^1 \beta(x, y)\, \alpha(y)\, dy,$$

$$\left(\int_0^1 \beta_1(x, y)\, \eta(y)\, dy; \int_0^1 \beta_2(x, y)\, \eta(y)\, dy\right)$$

$$= \int_0^1 \int_0^1 [\beta_1(x, u)\, \beta_2(u, y) - \beta_2(x, u)\, \beta_1(u, y)\, du]\, \eta(y)\, dy$$

are all of the form $\alpha(x) + \int_0^1 \beta(x, y)\, \eta(y)\, dy$ in view of the fact that

$$\gamma(x, y) = \int_0^1 \{\beta_1(x, u)\, \beta_2(u, y) - \beta_2(x, u)\, \beta_1(u, y)\}\, du$$

satisfies the condition

$$\gamma(x, y) + \gamma(y, x) = 0.$$

5. **The group of motions in function space.** By analogy with the situation in n-space let the group of motions in function space be defined as all the regular infinitesimal transformations which leave invariant

$$\int_0^1 [\alpha(x) - \beta(x)]^2\, dx \tag{21}$$

where $\alpha(x)$ and $\beta(x)$ are any two continuous functions on the interval I. It will be seen that all these transformations do form a group in the sense defined in the preceding section. One must have

$$\delta \int_0^1 [\alpha(x) - \beta(x)]^2\, dx = 0$$

or

$$\int_0^1 \alpha(x)\, \delta\alpha(x)\, dx + \int_0^1 \beta(x)\, \delta\beta(x)\, dx$$

$$- \int_0^1 \alpha(x)\, \beta(x)\, dx \left[\int_0^1 \alpha(x)\, \delta\beta(x)\, dx + \int_0^1 \beta(x)\, \delta\alpha(x)\, dx\right] = 0$$

must be an identity for all continuous functions α, β. Or, by (5)

$$\int_0^1 \alpha(x)\,\mathfrak{P}(x, \varphi; \alpha)\,dx + \int_0^1 \beta(x)\,\mathfrak{P}(x, \varphi; \beta)\,dx$$
$$- \int_0^1 \alpha(x)\beta(x)\,dx\left[\int_0^1 \alpha(x)\,\mathfrak{P}(x, \varphi; \beta)\,dx + \int_0^1 \beta(x)\,\mathfrak{P}(x, \varphi; \alpha)\,dx\right] = 0.$$

In particular, if $\alpha(x)$ and $\beta(x)$ satisfy $\int_0^1 \alpha^2(x)\,dx = \int_0^1 \beta^2(x)\,dx$, $\int_0^1 \alpha(x)\,\beta(x)\,dx = 0$, this last condition is seen to be precisely (7). Since it may readily be shown that (17) leaves (21) invariant one has proved the

THEOREM. *The group of motions in function space is given by*

$$\delta\varphi(x) = \left[\alpha(x) + \int_0^1 \beta(x, y)\,\varphi(y)\,dy\right]\delta t \tag{17}$$

where $\alpha(x)$ is an arbitrary continuous function of x and

$$\beta(x, y) = -\beta(y, x).$$

It is thus seen that this group is the same as the area preserving group.

One can show that the transformations behave like the group of motions in n-space in that they leave invariant the curvatures of that space.

The curvatures in a function space are defined as follows.* Consider a function $\varphi(x, s)$ continuous in x and continuous and possessing continuous derivatives of all orders with respect to s. This function defines a curve in the space of infinitely many dimensions just as $\varphi_i(s)$, $(i = 1, 2, \ldots n)$, defines a curve in n-space. Let it further be assumed that the parameter s is so chosen that it satisfies

$$\int_0^1 \left(\frac{\partial\varphi(x, s)}{\partial s}\right)^2 dx = 1.$$

This can always be done as Ingold has shown.† Let $k_i(s)$ denote the ith curvature of $\varphi(x, s)$. Then Ingold has defined‡ the k_i curvatures as follows:

* Kowalewski, Les Formules de Frenet dans l'espace fonctional, Comptes Rendus, vol. 151 (1910), p. 1338. Ingold, Differential Geometry in a Function Space, Trans. Amer. Math. Soc., vol. 13 (1912), p. 319.

† Loc. cit., p. 320.

‡ Loc. cit., p. 321.

$$\begin{cases} k_1^2(s) = \dfrac{1}{r_1^2(s)} = \displaystyle\int_0^1 \left(\frac{\partial n_0(x,s)}{\partial s} \right)^2 dx \\ k_i^2(s) = \dfrac{1}{r_i^2(s)} = \displaystyle\int_0^1 \left(\frac{\partial n_{i-1}(x,s)}{\partial s} + \frac{n_{i-2}(x,s)}{r_{i-1}} \right)^2 dx \end{cases}$$

where

$$\begin{cases} n_0(x,s) = \dfrac{\partial \varphi(x,s)}{\partial s}, \qquad \dfrac{n_1(x,s)}{r_1(s)} = \dfrac{\partial n_0(x,s)}{\partial s} \\ \dfrac{n_i(x,s)}{r_i(s)} = \dfrac{\partial n_{i-1}(x,s)}{\partial s} + \dfrac{n_{i-2}(x,s)}{r_{i-2}}. \end{cases}$$

From these defining relations he is able to prove that the k_i are given by the formulas*

$$k_j^2 = r_1^{2j}\, r_2^{2(j-1)} \cdots r_{j-1}^4 \int_0^1 \cdots \int_0^1 \varphi_s^2(x,s)\, \varphi_{ss}^2(x_1,s)\, \varphi_{sss}^2(x_2,s) \cdots \cdots \varphi_{sss\ldots}^2(x,s)\, dx\, dx_1 \cdots dx_j$$

$$= r_1^{2j}\, r_2^{2j-2} \cdots r_{j-1}^4 \int_0^1 \varphi_{ss}^2 \int_0^1 \varphi_{sss}^2 \cdots \int_0^1 \varphi_{ss\cdots j+1\text{ times}}^2$$

since $\int_0^1 \varphi_s^2 = 1$. The notation $\varphi_{ss\cdots j+1\text{ times}}$ means $\dfrac{\partial^{j+1}\varphi(x,s)}{\partial s^{j+1}}$, and the variables of integration are omitted in the last expression.

From this, one easily derives the following

$$k_j^2 = r_1^2\, r_2^2 \cdots r_{j-1}^2 \int_0^1 \varphi_{ss\cdots j+1\text{ times}}^2. \tag{22}$$

Since $k_1 = \int_0^1 \varphi_{ss}^2$ one has

$$k_1\, \delta k_1 = \int \varphi_{ss}\, \delta \varphi_{ss}. \tag{23}$$

Furthermore, if in the transformation

$$\delta \varphi(x) = \left[\alpha(x) + \int_0^1 \beta(x,y)\, \varphi(y)\, dy \right] \delta t$$

* Loc. cit., p. 324.

one supposes $\varphi(x)$ replaced by $\varphi(x, s)$, one gets

$$\delta\varphi(x, s) = \left[\alpha(x) + \int_0^1 \beta(x, y)\, \varphi(y, s)\, dy\right] \delta t.$$

By repeated differentiation with respect to s, one obtains

$$\delta\varphi_{ss\cdots j \text{ times}} = \int_0^1 \beta(x, y)\, \varphi_{ss\cdots j \text{ times}}(y, s)\, dy. \tag{24}$$

Substituting this in (23) one readily sees that $\delta k_1 = 0$ since $\beta(x,y) + \beta(y,x) = 0$. Suppose one has proved that $\delta k_{j-1} = 0$ $(j = 2, \ldots m)$. This carries with it that $\delta r_{j-1} = 0$ $(j = 2, \ldots m)$. By (22)

$$k_m^2 = r_1^2 r_2^2 \cdots r_{m-1}^2 \int_0^1 \varphi^2_{ss\cdots m+1 \text{ times}}.$$

Hence

$$2k_m \delta k_m = r_1^2 r_2^2 \cdots r_{m-1}^2 \int_0^1 \varphi_{ss\cdots m+1 \text{ times}}\, \delta\varphi_{ss\cdots m+1 \text{ times}} + \int_0^1 \varphi^2_{ss\cdots m+1 \text{ times}}$$

[terms involving a linear homogeneous combination of $\delta r_1 \cdots \delta r_{m-1}$].

Substituting (24) in this, one finds readily that $\delta k_m = 0$ which completes the induction.

6. **An extension.** Since $\int_0^1 \alpha^2(x)\, dx \int_0^1 \beta^2(x)\, dx - \left(\int_0^1 \alpha(x)\, \beta(x)\, dx\right)^2$ may be written as a Gramian determinant

$$\begin{vmatrix} \int_0^1 \alpha^2(x)\, dx & \int_0^1 \alpha(x)\, \beta(x)\, dx \\ \int_0^1 \alpha(x)\, \beta(x)\, dx & \int_0^1 \beta^2(x)\, dx \end{vmatrix}$$

one is naturally led to inquire into the most general class of regular infinitesimal transformations which leave the Gramian

$$\begin{vmatrix} \int_0^1 \alpha_1^2(x)\, dx \cdots \int_0^1 \alpha_1(x)\, \alpha_p(x)\, dx \\ \cdot \\ \cdot \\ \cdot \\ \int_0^1 \alpha_p(x)\, \alpha_1(x)\, dx \cdots \int_0^1 \alpha_p^2(x)\, dx \end{vmatrix} \tag{25}$$

invariant. By the methods used in § 2, one readily finds that a necessary condition is that

$$\int_0^1 \alpha_1(x)\,\mathfrak{P}(x, \varphi; \alpha_1)\,dx + \int_0^1 \alpha_2(x)\,\mathfrak{P}(x, \varphi; \alpha_2)\,dx + \cdots \cdots + \int_0^1 \alpha_p(x)\,\mathfrak{P}(x, \varphi; \alpha_p)\,dx = 0$$

for all α_i's for which

$$\int_0^1 \alpha_i(x)\,\alpha_j(x)\,dx = 0, \quad i \neq j, \qquad \int_0^1 \alpha_i^2(x)\,dx = \int_0^1 \alpha_j^2(x)\,dx.$$

It is not hard to show (cf. § 3) that this condition leads one to the same group as the special case $p = 2$, viz.,

$$\delta\varphi = \left[\alpha(x) + \int_0^1 \beta(x, y)\,\varphi(y)\,dy\right]\delta t$$

and so, in conclusion, one has the theorem

The most general regular infinitesimal transformation which leaves invariant the Gramian (25) *where p is any given integer, is the same group as the group of motions in function space.*

CONGRUENCES OF CIRCLES STUDIED WITH REFERENCE TO THE SURFACE OF CENTERS.

By J. M. Thomas.

1. **Introduction.** Two parameter systems of circles which possess a single infinity of orthogonal surfaces have been extensively studied, especially by Ribaucour,* who named them cyclic systems. In particular, Ribaucour proved the following theorem: if in the tangent planes to a pseudospherical surface of curvature $-1/a^2$ circles of radius a be described with centers at the points of contact, these circles are the orthogonal trajectories of an infinity of surfaces of curvature $-1/a^2$. The process by which one of the orthogonal surfaces is obtained from the original pseudospherical surface has been called the transformation of Bianchi. This transformation has been extended by Bäcklund† to the case where the circles are cut at any constant angle by their trajectories. Beal‡ has further extended the transformation so that the angle is a function of the curvilinear coördinates on the original surface. The curvature is the same at corresponding points of the original and transformed surfaces. It satisfies a differential equation of the second order.

In all of the work mentioned above the framework of reference is the surface of centers only when the surface of centers coincides with the envelope of the planes of the circles. In the present paper are discussed certain properties possessed by a two parameter system of circles in connection with its surface of centers S, when the latter is distinct from the envelope of the planes. The lines of intersection of the planes of the circles with the tangent planes to S form a rectilinear congruence L. Since S is a focal surface of L, the edges of regression of one family of developable surfaces of L lie on S. It is therefore always possible to assemble the lines of L so that they are tangent to a single parameter family of curves Γ on S.

In § 2 the reference scheme to be used in this paper is explained. In § 3 certain equations of condition connected with the congruence of axes and the envelope of the planes are derived, and two theorems are proved

* "Mémoire sur la théorie générale des surfaces courbes", Journ. de Math., Ser. 4, vol. VII (1891), Chap. XI, XIII.

† Cf. Eisenhart, Differential Geometry, p. 284. Further reference to this text will be made in the form Eisenhart, p. 284.

‡ These Annals, 2d ser., vol. 17 (1916), p. 183 et seq.

to show interdependence among the congruence of axes, the curves Γ and the angle ψ between the plane of the circle and the normal to S.

The congruence of circles is next restricted so that each point P of a circle is on a surface S_1 which is one of a single parameter family of surfaces of the following nature. The tangent plane to S_1 passes through the center of the circle and makes an angle $90^\circ - \Phi$ with the plane of the circle, Φ being a function of u and v, the curvilinear coördinates on S. Beal* has pointed out the existence of such congruences for a given surface of centers, and has studied in detail the case where the circles are in the tangent planes to S, that is, $\psi = 90^\circ$. We call congruences of the above type for which ψ is not a right angle congruences C. In § 4 the fundamental equations for the existence of such congruences are derived and discussed.

In § 5 is proved the existence of congruences C in which all the circles have the same radius and the angle Φ is constant. Geometric interpretations of the three fundamental conditions are given for this case. It is found, for instance, that the circles must lie in the osculating planes of the curves Γ.

The cases where S degenerates into a curve or a point are discussed in § 6 and § 7 respectively.

The writer is greatly indebted to Professor F. W. Beal for assistance in the preparation of this paper.

2. With S we associate a set of moving rectangular axes, the x and y axes lying in the tangent plane and the z axis being the normal. The total displacements of a point (x, y, z) with respect to the moving axes are†

$$(1)\qquad \begin{cases} \delta x = dx + \xi\,du + \xi_1\,dv + (q\,du + q_1\,dv)z - (r\,du + r_1\,dv)y, \\ \delta y = dy + \eta\,du + \eta_1\,dv + (r\,du + r_1\,dv)x - (p\,du + p_1\,dv)z, \\ \delta z = dz \qquad\qquad + (p\,du + p_1\,dv)y - (q\,du + q_1\,dv)x. \end{cases}$$

In these formulas the translations ξ, ξ_1, η, η_1, and the rotations p, q, r, p_1, q_1, r_1 must satisfy the following conditions‡ which are equivalent to the Gauss and Codazzi equations for S:

$$(2)\qquad \begin{cases} p_v - p_{1u} = q r_1 - q_1 r, & \xi_v - \xi_{1u} = \eta r_1 - \eta_1 r, \\ q_v - q_{1u} = r p_1 - r_1 p, & \eta_v - \eta_{1u} = \xi_1 r - \xi r_1, \\ r_v - r_{1u} = p q_1 - p_1 q, & \xi q_1 - \xi_1 q = \eta p_1 - \eta_1 p. \end{cases}$$

* L. c., p. 182.

† Eisenhart, p. 170.

‡ Eisenhart, pp. 168, 170. Here, as throughout this paper, the subscripts u, v denote partial derivatives. Thus $p_v = \partial p/\partial v$.

If the x axis is chosen tangent to the curve $v = \text{const.}$ and an orthogonal system is parametric on S, then $\xi_1 = \eta = 0$, and the rotations have the following values*

$$(3)\quad \begin{cases} p = \bar{\omega}_u - \xi/\tau, & q = -\xi\cos\bar{\omega}/\varrho, & r = \xi\sin\bar{\omega}/\varrho, \\ p_1 = \eta_1\cos\bar{\omega}_1/\varrho_1, & q_1 = \bar{\omega}_{1v} - \eta_1/\tau_1, & r_1 = \eta_1\sin\bar{\omega}_1/\varrho_1, \end{cases}$$

where $1/\varrho$ is the curvature, $1/\tau$ the torsion and $\bar{\omega}$ the angle between the normal to S and the principal normal of $v = \text{const.}$, and the letters with the subscript 1 denote corresponding quantities for $u = \text{const.}$

2. **Congruences in general.** Taking $\xi_1 = \eta = 0$ and the curves l as $v = \text{const.}$, we next deduce certain properties of congruences in general. The axes of the circles form a rectilinear congruence. The coördinates of a point Q on the axis of a circle at distance t from its center are $(0, -t\cos\psi, t\sin\psi)$, and the direction cosines of the axis are proportional to these coördinates. To find the developables of the congruence of axes we express the condition that Q move tangent to the axis. It is

$$(4)\quad \frac{\delta x}{0} = \frac{\delta y}{-\cos\psi} = \frac{\delta z}{\sin\psi}.$$

Making use of (1) we can replace (4) by the equivalent conditions

$$(5)\quad \begin{cases} \xi\,du + [(q\,du + q_1\,dv)\sin\psi + (r\,du + r_1\,dv)\cos\psi]\,t = 0, \\ \eta_1\sin\psi\,dv + (d\psi - p\,du - p_1\,dv)\,t = 0. \end{cases}$$

Elimination of t gives as the differential equation of the developables,

$$(6)\quad \xi(\psi_u - p)\,du^2 + (\xi\psi_v - \xi p_1 - q\eta_1\sin^2\psi - r\eta_1\sin\psi\cos\psi)\,du\,dv - \eta_1\sin\psi(q_1\sin\psi + r_1\cos\psi)\,dv^2 = 0.$$

The condition† that these correspond to a conjugate system on S can be reduced to

$$(7)\quad p_1\psi_u - p\psi_v + (qr_1 - q_1 r)\sin\psi\cos\psi = 0.$$

Making use of the first of (2) and dividing by $\sin^2\psi$ we are led to

$$(8)\quad \frac{\partial}{\partial v}(p\cot\psi) = \frac{\partial}{\partial u}(p_1\cot\psi).$$

* Eisenhart, p. 174.

† Eisenhart, pp. 130, 174.

The curves conjugate to $u = \text{const.}$ have for differential equation*

$$p\,du + p_1\,dv = 0.$$

Equation (8) expresses that $\cot\psi$ is an integrating factor of this last equation. Hence when (8) is true, the curves conjugate to the orthogonal trajectories of Γ can be found by a quadrature. Conversely, if the finite equation of these curves is known to be

$$f(u, v) = 0,$$

then the angle ψ can be determined so that (8) is satisfied by taking

$$\cot\psi = \frac{1}{p}\frac{\partial f}{\partial u} = \frac{1}{p_1}\frac{\partial f}{\partial v}.$$

If the lines of curvature are parametric, $p = q_1 = 0$ and (8) can be written

$$p_1 \cot\psi = V, \tag{9}$$

where V is a function of v alone. If now $r_1 = 0$, it is seen from (3) that the curves $u = \text{const.}$ are geodesics. Being also lines of curvature, they are plane. From the first equation of (2) it follows that p_1 is a function of v alone and hence from (9) that ψ is likewise independent of u. Reasoning from the same equations in a similar fashion we find we can state the following

THEOREM 1. *If the curves Γ are lines of curvature and S is not developable, each of the following statements is a consequence of the remaining two:*

(i) *The developables of the congruence of axes correspond to a conjugate system on S.*

(ii) *The orthogonal trajectories of the curves Γ are plane geodesics.*

(iii) *The inclination of the plane of the circle to the normal of S is constant along each curve Γ.*

By eliminating $du : dv$ from equations (5) we obtain

$$\begin{aligned}\xi\eta_1 \sin\psi + (\xi\psi_v - \xi p_1 + q\eta_1 \sin^2\psi + r\eta_1 \sin\psi\cos\psi)\,t \\ + [(q\psi_v - q_1\psi_u + pq_1 - p_1 q)\sin\psi \\ + (r\psi_v - r_1\psi_u - rp_1 + r_1 p)\cos\psi]\,t^2 = 0.\end{aligned} \tag{10}$$

The roots t_1, t_2 of this equation are the distances to the focal points.

* Eisenhart, p. 173.

If on the other hand Q is to move normal to the axis, we must have

$$\delta y \cos\psi - \delta z \sin\psi = 0,$$

or

$$dt = \eta_1 \cos\psi \, dv.$$

The condition that the congruence of axes be normal is therefore

$$\eta_1 \cos\psi = V, \tag{11}$$

where V denotes a function of v alone. If $r_1 = 0$, it follows from (2) that η_1 is independent of u and from (11) that the same is true of ψ. We can in this manner prove

THEOREM 2. *Each of the following statements is a consequence of the other two:*

(i) *The congruence of axes of the circles is normal.*

(ii) *The orthogonal trajectories of the curves Γ are geodesics.*

(iii) *The inclination of the plane of the circle to the normal of S is constant along each curve Γ.*

We next find the coördinates (x_0, y_0, z_0) of the point of contact of the plane of the circle with its envelope. The condition on the coördinates is

$$\delta y_0 \cos\psi - \delta z_0 \sin\psi = 0.$$

Putting

$$y_0 = \lambda \sin\psi, \qquad z_0 = \lambda \cos\psi,$$

substituting the values of δy_0 and δz_0 from (1) and equating the coefficients of du and dv to zero we get

$$\begin{aligned} (q \sin\psi + r \cos\psi) x_0 + (\psi_u - p)\lambda &= 0, \\ (q_1 \sin\psi + r_1 \cos\psi) x_0 + (\psi_v - p_1)\lambda &= -\eta_1 \cos\psi. \end{aligned}$$

If we let $\mu = 1/t_1 t_2$ and make use of (10) we can put the solution of these equations in the form

$$x_0 = (\psi_u - p) \cot\psi/\xi\mu, \qquad \lambda = -(q \sin\psi + r \cos\psi) \cot\psi/\xi\mu. \tag{12}$$

3. **Congruences C.** If R denotes the radius of a circle and $\varkappa$ the angle made by the corresponding line of L with the x axis of the trihedral, the coördinates of a point P on the circle are

(13) $R(\cos\theta\cos\varkappa - \sin\theta\sin\psi\sin\varkappa)$, $R(\cos\theta\sin\varkappa + \sin\theta\sin\psi\cos\varkappa)$, $R\sin\theta\cos\psi$,

where θ denotes the angle made by the radius to P with the line L. The condition that P generate a surface S_1, as defined in the introduction, is

$$l\delta x + m\delta y + n\delta z = 0, \tag{14}$$

where l, m, n are the direction cosines of the normal to S_1 and have the values

$$\begin{cases} l = \sin\theta\cos\Phi\cos\varkappa + (\sin\Phi\cos\psi + \cos\theta\cos\Phi\sin\psi)\sin\varkappa, \\ m = \sin\theta\cos\Phi\sin\varkappa - (\sin\Phi\cos\psi + \cos\theta\cos\Phi\sin\psi)\cos\varkappa, \\ n = \sin\Phi\sin\psi - \cos\theta\cos\Phi\cos\psi. \end{cases} \tag{15}$$

When substitutions are made from (1), (13) and (15), equation (14) becomes

$$A\,d\theta + B\,du + C\,dv = 0, \tag{16}$$

where

$$\begin{aligned} A &= -R\cos\Phi, \\ B &= (\xi\cos\varkappa + \eta\sin\varkappa)\sin\theta\cos\Phi \\ &\quad + (\xi\sin\varkappa - \eta\cos\varkappa)(\cos\theta\cos\Phi\sin\psi + \sin\Phi\cos\psi) \\ &\quad + R(p\cos\varkappa + q\sin\varkappa - \psi_u)\sin\theta\sin\Phi \\ &\quad + R(p\sin\varkappa - q\cos\varkappa)(\cos\theta\sin\Phi\sin\psi - \cos\Phi\cos\psi) \\ &\quad + R(r + \varkappa_u)(\cos\theta\sin\Phi\cos\psi + \cos\Phi\sin\psi), \end{aligned}$$

and C can be obtained from B by adding the subscript 1 to the translations and rotations and changing the subscript u into v. Equation (16) can be made to assume the Riccati form by the substitution $\theta = 2\tan^{-1}t$.

The necessary and sufficient condition for a congruence C is that equation (16) possess a solution $\theta(u, v)$ involving an arbitrary parameter, or that the condition of integrability of (16), namely,

$$A\left(\frac{\partial C}{\partial u} - \frac{\partial B}{\partial v}\right) + B\left(\frac{\partial A}{\partial v} - \frac{\partial C}{\partial \theta}\right) + C\left(\frac{\partial B}{\partial \theta} - \frac{\partial A}{\partial u}\right) = 0,$$

be identically satisfied. This condition assumes the form

$$L\sin\theta + M\cos\theta + N = 0,$$

where L, M, and N are independent of θ. The equation in θ is satisfied by not more than two values unless $L = M = N = 0$, when it is identically satisfied. The fundamental equations for the existence of a congruence C are accordingly

$$
\begin{aligned}
(17)\quad & R(\xi p_1 - \xi_1 p)\sin\psi\cos\psi\sin^2\varkappa - R(\xi q_1 - \xi_1 q)\sin 2\psi\sin\varkappa\cos\varkappa \\
& - R(\xi r_1 - \xi_1 r)\cos^2\psi\sin\varkappa + R(\eta q_1 - \eta_1 q)\sin\psi\cos\psi\cos^2\varkappa \\
& + R(\eta r_1 - \eta_1 r)\cos^2\psi\cos\varkappa - (\xi R_v - \xi_1 R_u)\cos^2\Phi\cos\varkappa \\
& - (\eta R_v - \eta_1 R_u)\cos^2\Phi\sin\varkappa + R^2(\psi_v\Phi_u - \psi_u\Phi_v) \\
& + R^2(p\Phi_v - p_1\Phi_u)\cos\varkappa + R^2(q\Phi_v - q_1\Phi_u)\sin\varkappa \\
& - R(\xi\varkappa_v - \xi_1\varkappa_u)\cos^2\psi\sin\varkappa + R(\eta\varkappa_v - \eta_1\varkappa_u)\cos^2\psi\cos\varkappa = 0,
\end{aligned}
$$

$$
\begin{aligned}
(18)\quad & (\xi\eta_1 - \xi_1\eta)\sin\Phi\cos\Phi\cos\psi + R(\xi p_1 - \xi_1 p)\cos\psi\sin\varkappa\cos\varkappa \\
& - R(\xi q_1 - \xi_1 q)\cos\psi\cos 2\varkappa - R(\eta q_1 - \eta_1 q)\cos\psi\sin\varkappa\cos\varkappa \\
& + (\xi R_v - \xi_1 R_u)\cos^2\Phi\sin\psi\sin\varkappa - (\eta R_v - \eta_1 R_u)\cos^2\Phi\sin\psi\cos\varkappa \\
& - R(\xi\psi_v - \xi_1\psi_u)\cos\psi\sin\varkappa + R(\eta\psi_v - \eta_1\psi_u)\cos\psi\cos\varkappa \\
& - R^2(p\Phi_v - p_1\Phi_u)\sin\psi\sin\varkappa + R^2(q\Phi_v - q_1\Phi_u)\sin\psi\cos\varkappa \\
& + R^2(r\Phi_v - r_1\Phi_u)\cos\psi + R^2(\Phi_v\varkappa_u - \Phi_u\varkappa_v)\cos\psi = 0,
\end{aligned}
$$

$$
\begin{aligned}
(19)\quad & (\xi\eta_1 - \xi_1\eta)\cos^2\Phi\sin\psi + R^2(qr_1 - q_1 r)\cos\psi\sin\varkappa \\
& - R^2(rp_1 - r_1 p)\cos\psi\cos\varkappa + R^2(pq_1 - p_1 q)\sin\psi \\
& + (\xi R_v - \xi_1 R_u)\sin\Phi\cos\Phi\cos\psi\sin\varkappa \\
& - (\eta R_v - \eta_1 R_u)\sin\Phi\cos\Phi\cos\psi\cos\varkappa - R^2(p\psi_v - p_1\psi_u)\sin\psi\sin\varkappa \\
& + R^2(q\psi_v - q_1\psi_u)\sin\psi\cos\varkappa + R^2(r\psi_v - r_1\psi_u)\cos\psi \\
& + R^2(\psi_v\varkappa_u - \psi_u\varkappa_v)\cos\psi - R(\xi\Phi_v - \xi_1\Phi_u)\cos\psi\sin\varkappa \\
& + R(\eta\Phi_v - \eta_1\Phi_u)\cos\psi\cos\varkappa + R^2(p\varkappa_v - p_1\varkappa_u)\cos\psi\cos\varkappa \\
& + R^2(q\varkappa_v - q_1\varkappa_u)\cos\psi\sin\varkappa = 0.
\end{aligned}
$$

In discussing this system of equations we can without loss of generality put $\xi_1 = \eta = p = q_1 = 0$. This amounts to taking the lines of curvature parametric on S and the coördinate axes tangent to them. Since the equations can be solved for the u derivatives of three of the four quantities R, Φ, ψ and $\varkappa$ the following statements are true in general:

(i) Any one of the functions R, Φ, ψ and $\varkappa$, or one relation among them, can be chosen arbitrarily. Thus, choosing $\Phi = 0$, we see that any sur-

(13) $R(\cos\theta\cos\varkappa - \sin\theta\sin\psi\sin\varkappa)$, $R(\cos\theta\sin\varkappa + \sin\theta\sin\psi\cos\varkappa)$, $R\sin\theta\cos\psi$,

where θ denotes the angle made by the radius to P with the line L. The condition that P generate a surface S_1, as defined in the introduction, is

$$l\delta x + m\delta y + n\delta z = 0, \tag{14}$$

where l, m, n are the direction cosines of the normal to S_1 and have the values

$$\begin{cases} l = \sin\theta\cos\Phi\cos\varkappa + (\sin\Phi\cos\psi + \cos\theta\cos\Phi\sin\psi)\sin\varkappa, \\ m = \sin\theta\cos\Phi\sin\varkappa - (\sin\Phi\cos\psi + \cos\theta\cos\Phi\sin\psi)\cos\varkappa, \\ n = \sin\Phi\sin\psi - \cos\theta\cos\Phi\cos\psi. \end{cases} \tag{15}$$

When substitutions are made from (1), (13) and (15), equation (14) becomes

$$A\,d\theta + B\,du + C\,dv = 0, \tag{16}$$

where

$$\begin{aligned} A &= -R\cos\Phi, \\ B &= (\xi\cos\varkappa + \eta\sin\varkappa)\sin\theta\cos\Phi \\ &\quad + (\xi\sin\varkappa - \eta\cos\varkappa)(\cos\theta\cos\Phi\sin\psi + \sin\Phi\cos\psi) \\ &\quad + R(p\cos\varkappa + q\sin\varkappa - \psi_u)\sin\theta\sin\Phi \\ &\quad + R(p\sin\varkappa - q\cos\varkappa)(\cos\theta\sin\Phi\sin\psi - \cos\Phi\cos\psi) \\ &\quad + R(r + \varkappa_u)(\cos\theta\sin\Phi\cos\psi + \cos\Phi\sin\psi), \end{aligned}$$

and C can be obtained from B by adding the subscript 1 to the translations and rotations and changing the subscript u into v. Equation (16) can be made to assume the Riccati form by the substitution $\theta = 2\tan^{-1}t$.

The necessary and sufficient condition for a congruence C is that equation (16) possess a solution $\theta(u, v)$ involving an arbitrary parameter, or that the condition of integrability of (16), namely,

$$A\left(\frac{\partial C}{\partial u} - \frac{\partial B}{\partial v}\right) + B\left(\frac{\partial A}{\partial v} - \frac{\partial C}{\partial \theta}\right) + C\left(\frac{\partial B}{\partial \theta} - \frac{\partial A}{\partial u}\right) = 0,$$

be identically satisfied. This condition assumes the form

$$L\sin\theta + M\cos\theta + N = 0,$$

where L, M, and N are independent of θ. The equation in θ is satisfied by not more than two values unless $L = M = N = 0$, when it is identically satisfied. The fundamental equations for the existence of a congruence C are accordingly

$$
\begin{aligned}
(17)\quad & R(\xi p_1 - \xi_1 p)\sin\psi\cos\psi\sin^2\varkappa - R(\xi q_1 - \xi_1 q)\sin 2\psi\sin\varkappa\cos\varkappa \\
& - R(\xi r_1 - \xi_1 r)\cos^2\psi\sin\varkappa + R(\eta q_1 - \eta_1 q)\sin\psi\cos\psi\cos^2\varkappa \\
& + R(\eta r_1 - \eta_1 r)\cos^2\psi\cos\varkappa - (\xi R_v - \xi_1 R_u)\cos^2\Phi\cos\varkappa \\
& - (\eta R_v - \eta_1 R_u)\cos^2\Phi\sin\varkappa + R^2(\psi_v\Phi_u - \psi_u\Phi_v) \\
& + R^2(p\Phi_v - p_1\Phi_u)\cos\varkappa + R^2(q\Phi_v - q_1\Phi_u)\sin\varkappa \\
& - R(\xi\varkappa_v - \xi_1\varkappa_u)\cos^2\psi\sin\varkappa + R(\eta\varkappa_v - \eta_1\varkappa_u)\cos^2\psi\cos\varkappa = 0,
\end{aligned}
$$

$$
\begin{aligned}
(18)\quad & (\xi\eta_1 - \xi_1\eta)\sin\Phi\cos\Phi\cos\psi + R(\xi p_1 - \xi_1 p)\cos\psi\sin\varkappa\cos\varkappa \\
& - R(\xi q_1 - \xi_1 q)\cos\psi\cos 2\varkappa - R(\eta q_1 - \eta_1 q)\cos\psi\sin\varkappa\cos\varkappa \\
& + (\xi R_v - \xi_1 R_u)\cos^2\Phi\sin\psi\sin\varkappa - (\eta R_v - \eta_1 R_u)\cos^2\Phi\sin\psi\cos\varkappa \\
& - R(\xi\psi_v - \xi_1\psi_u)\cos\psi\sin\varkappa + R(\eta\psi_v - \eta_1\psi_u)\cos\psi\cos\varkappa \\
& - R^2(p\Phi_v - p_1\Phi_u)\sin\psi\sin\varkappa + R^2(q\Phi_v - q_1\Phi_u)\sin\psi\cos\varkappa \\
& + R^2(r\Phi_v - r_1\Phi_u)\cos\psi + R^2(\Phi_v\varkappa_u - \Phi_u\varkappa_v)\cos\psi = 0,
\end{aligned}
$$

$$
\begin{aligned}
(19)\quad & (\xi\eta_1 - \xi_1\eta)\cos^2\Phi\sin\psi + R^2(qr_1 - q_1 r)\cos\psi\sin\varkappa \\
& - R^2(rp_1 - r_1 p)\cos\psi\cos\varkappa + R^2(pq_1 - p_1 q)\sin\psi \\
& + (\xi R_v - \xi_1 R_u)\sin\Phi\cos\Phi\cos\psi\sin\varkappa \\
& - (\eta R_v - \eta_1 R_u)\sin\Phi\cos\Phi\cos\psi\cos\varkappa - R^2(p\psi_v - p_1\psi_u)\sin\psi\sin\varkappa \\
& + R^2(q\psi_v - q_1\psi_u)\sin\psi\cos\varkappa + R^2(r\psi_v - r_1\psi_u)\cos\psi \\
& + R^2(\psi_v\varkappa_u - \psi_u\varkappa_v)\cos\psi - R(\xi\Phi_v - \xi_1\Phi_u)\cos\psi\sin\varkappa \\
& + R(\eta\Phi_v - \eta_1\Phi_u)\cos\psi\cos\varkappa + R^2(p\varkappa_v - p_1\varkappa_u)\cos\psi\cos\varkappa \\
& + R^2(q\varkappa_v - q_1\varkappa_u)\cos\psi\sin\varkappa = 0.
\end{aligned}
$$

In discussing this system of equations we can without loss of generality put $\xi_1 = \eta = p = q_1 = 0$. This amounts to taking the lines of curvature parametric on S and the coördinate axes tangent to them. Since the equations can be solved for the u derivatives of three of the four quantities R, Φ, ψ and $\varkappa$ the following statements are true in general:

(i) Any one of the functions R, Φ, ψ and $\varkappa$, or one relation among them, can be chosen arbitrarily. Thus, choosing $\Phi = 0$, we see that any sur-

face can be the surface of centers of a cyclic system. Similarly the geometric property given by any one of equations (8), (9), (11), (27), etc., can be imposed without restricting S.

(ii) The function $\varkappa$ determines the family of curves Γ when the moving trihedral has been chosen. For any family of curves Γ on any surface S there exist functions R, Φ and ψ which associate with S a congruence C.

Equations (17), (18) and (19) become for $\varkappa = 0$

$$(20)\quad R^2(\psi_u \Phi_v - \psi_v \Phi_u) - R^2(p\,\Phi_v - p_1\,\Phi_u) - R\cos^2\psi(\eta\, r_1 - \eta_1 r) - R\sin\psi\cos\psi(\eta\, q_1 - \eta_1 q) + \cos^2\Phi(\xi\, R_v - \xi_1 R_u) = 0,$$

$$(21)\quad R^2\sin\psi(q\,\Phi_v - q_1\,\Phi_u) + R^2\cos\psi(r\,\Phi_v - r_1\,\Phi_u) + R\cos\psi(\eta\,\psi_v - \eta_1\,\psi_u) - R\cos\psi(\xi q_1 - \xi_1 q) + \sin\Phi\cos\Phi\cos\psi(\xi\,\eta_1 - \xi_1\,\eta) - \cos^2\Phi\sin\psi(\eta\, R_v - \eta_1 R_u) = 0,$$

$$(22)\quad R^2\sin\psi(q\,\psi_v - q_1\,\psi_u) + R^2\cos\psi(r\,\psi_v - r_1\,\psi_u) + R^2\sin\psi(p\,q_1 - p_1 q) - R^2\cos\psi(r\,p_1 - r_1\,p) + R\cos\psi(\eta\,\Phi_v - \eta_1\,\Phi_u) + \cos^2\Phi\sin\psi(\xi\,\eta_1 - \xi_1\,\eta) - \sin\Phi\cos\Phi\cos\psi(\eta\, R_v - \eta_1 R_u) = 0,$$

If R, Φ and ψ are given functions, then functions ξ, η, ξ_1, η_1, p, q, r, p_1, q_1, r_1 satisfying (2), (20), (21), and (22) can be shown to exist in general. For equations (20), (21), (22) and the last of (2) can be solved for ξ, η, p and q. If the results are substituted in equations (2), the latter can be solved for the u derivatives of p_1, q_1, r_1, ξ_1 and η_1 leaving r arbitrary, and the results are in a form to which Cauchy's theorem* can be applied to show the existence of the functions involved. Hence solutions, and consequently surfaces S, exist in general, but in particular cases a determinant involved in the solution may vanish and cause the method to fail. Moreover, even when the functions exist the corresponding surfaces S may be degenerate as is seen below in § 5 when R, Φ and ψ are all taken constant.

With $\varkappa$ still equal to zero, we next find the curvature of the surfaces S_1. If we take a point on the normal to S_1 at distance w from P, and express the condition that it move in the direction given by equations (15), we get two equations in $du : dv$. Elimination of this derivative gives the following equation to determine w:

$$(23)\qquad Pw^2 + Qw + T = 0,$$

* Jordan, Cours d'Analyse, (1896), t. III, p. 305.

where

$$\begin{aligned} P = {} & \sin\theta\cos^2\Phi\,[\Phi_v\xi - \Phi_u\xi_1 + (\psi_v\xi - \psi_u\xi_1)\cos\theta \\ & - (\xi q_1 - \xi_1 q)\sin\theta\sin\psi - (\xi p_1 - \xi_1 p)\cos\theta - (\xi r_1 - \xi_1 r)\sin\theta\cos\psi] \\ & - \cos\Phi(\sin\Phi\cos\psi + \cos\theta\cos\Phi\sin\psi)\,[\Phi_v\eta - \Phi_u\eta_1 \\ & + (\psi_v\eta - \psi_u\eta_1)\cos\theta - (\eta p_1 - \eta_1 p)\cos\theta \\ & - (\eta q_1 - \eta_1 q)\sin\theta\sin\psi - (\eta r_1 - \eta_1 r)\sin\theta\cos\psi], \end{aligned}$$

$$\begin{aligned} T = {} & - R^2\sin\theta\sin\psi\,[(\eta\psi_v - \eta_1\psi_u)\sin\theta - (\eta p_1 - \eta_1 p)\sin\theta \\ & + (\eta q_1 - \eta_1 q)\cos\theta\sin\psi + (\eta r_1 - \eta_1 r)\cos\theta\cos\psi] \\ & - R^2\cos\theta\,[(\xi\psi_v - \xi_1\psi_u)\sin\theta - (\xi p_1 - \xi_1 p)\sin\theta \\ & + (\xi q_1 - \xi_1 q)\cos\theta\sin\psi + (\xi r_1 - \xi_1 r)\cos\theta\cos\psi] \\ & - R(\xi\eta_1 - \xi_1\eta)\cos\theta\cos\psi - R^2\sin\theta(\psi_v R_u - \psi_u R_v) \\ & + R\cos\psi(\eta R_v - \eta_1 R_u) - R^2\sin\theta(p R_v - p_1 R_u) \\ & + R^2\cos\theta\sin\psi(q R_v - q_1 R_u) + R^2\cos\theta\cos\psi(r R_v - r_1 R_u), \end{aligned}$$

and Q does not contain w. The curvature K_1 of the surfaces S_1 is the reciprocal of the product of the roots of (23), that is

$$K_1 = P/T. \tag{24}$$

This expression is not in general independent of θ, but when $\psi = 90^0$, it reduces to

$$K_1 = -\cos^2\Phi/R^2,$$

and all of the transformed surfaces S_1 have the same total curvature as the original surface S.* The derivative of K_1 with respect to θ equated to zero yields an equation of the fourth degree in $\sin\theta$ and of the first in $\cos\theta$. This equation gives the necessary condition for transformed surfaces of maximum or minimum curvature at a point. It is not satisfied by more than eight independent values of θ unless it is identically satisfied, in which case the curvature is independent of θ.

THEOREM 3. *The curvature of the surfaces S_1 cannot have more than eight extreme values at a point of general position on S.*

The direction cosines of the planes normal to S_1 and passing through one of the corresponding focal points of the congruence of axes are readily found to be proportional to

* This result is given by Beal, loc., cit., p. 184.

$$t_i \cos\theta \cos\Phi - R \sin\theta \sin\Phi,$$

$$t_i \sin\theta \cos\Phi \sin\psi - R \cos\Phi \cos\psi + R \cos\theta \sin\Phi \sin\psi,$$

$$t_i \sin\theta \cos\Phi \cos\psi + R \cos\Phi \sin\psi + R \cos\theta \sin\Phi \cos\psi, \quad (i = 1, 2),$$

where the t's satisfy (10). The condition that the two planes be perpendicular can be reduced to

$$1/t_1 t_2 + \cos^2\Phi/R^2 = 0. \tag{25}$$

This condition remains the same whatever the value of $\varkappa$ or ψ. By the use of (10), equation (22) can be written in the form ($\xi_1 = \eta = 0$)

$$1/t_1 t_2 + \cos^2\Phi/R^2 = \cot\psi(R\Phi_u - R_u \sin\Phi\cos\Phi)/\xi R^2.$$

The necessary and sufficient condition that condition (25) be satisfied by a congruence C is therefore

$$\cot\psi(R\Phi_u - R_u \sin\Phi\cos\Phi) = 0. \tag{26}$$

When $\cot\psi$ is divided out* and the equation is integrated, (26) becomes

$$R\cot\Phi = V, \tag{27}$$

where V is a function of v alone. Conversely, if (27) is satisfied, (25) is also, unless $\psi = 0$. In the exceptional case one focal point is at the origin and the other is the point at infinity on the y axis. It is readily seen geometrically that the two planes in question are still perpendicular. The solution $\Phi = 0$ of (26) is not given by (27). It corresponds to a cyclic system, for which it is well known† that the joins of a point on the circle to the corresponding focal points, and consequently the above normal planes to S_1, are perpendicular. We have therefore

THEOREM 4. *For a congruence C which is not a cyclic system the necessary and sufficient condition for the perpendicularity of the two planes which are normal to S_1 and pass through the corresponding focal points of the congruence of axes is that $R\cot\Phi$ be constant along each curve Γ.*

* $\cot\psi \neq 0$ for a congruence C. If $\psi = 90^0$, equation (25) is satisfied. Beal, loc. cit. p. 183, has remarked that an equivalent geometric property is possessed by his congruences C_0.

† Eisenhart, p. 437.

In the case of cyclic systems, the formulas (12) can be written in a compact form. Equations (20) and (21) become for $\Phi = 0$, $\xi_1 = \eta = 0$

$$R_u/R = (\psi_u - p)\cot\psi, \qquad R_v/R = -\eta_1 \cos\psi\,(q\sin\psi + r\cos\psi)/\xi.$$

From (25) we have

$$\mu = -1/R^2.$$

Hence formulas (12) can be written

$$x_0 = -\frac{1}{2}\frac{dR^2}{ds}, \qquad y_0 = -\frac{1}{2}\frac{dR^2}{ds_1}, \qquad z_0 = -\frac{1}{2}\frac{dR^2}{ds_1}\cot\psi,$$

where s and s_1 denote the arc of the curves Γ and of their orthogonal trajectories respectively.

4. **Congruences $C_{R\Phi}$.** A congruence C for which R and Φ are constant* will be denoted by $C_{R\Phi}$, the subscripts indicating the quantities which are kept constant. The fundamental equations for such a congruence are ($\xi_1 = \eta = 0$)

$$\tan\psi = -r/q, \tag{28}$$

$$\psi_u = p + a\xi, \tag{29}$$

$$b\eta_1 = q_1 + r_1\cot\psi, \tag{30}$$

where a and b are constants defined by

$$a = \sin\Phi\cos\Phi/R, \tag{31}$$

$$b = \cot\Phi/R. \tag{32}$$

From (3) and (28) it is seen that ψ can be taken equal to $\bar{\omega}$. Hence the circles lie in the osculating planes of the curves Γ. Equation (29) can be rewritten as

$$\bar{\omega}_u = p + a\xi.$$

* The case $\psi = 90^\circ$ is, of course, excluded. If $\psi = 90^\circ$ and R and Φ are constant, S is a pseudospherical surface, as is seen from (25). The surfaces S_1 are its Bäcklund transforms. Cf. Beal, loc. cit., p. 185. Also, Eisenhart, p. 284.

By use of the first of equations (3) we get $1/\tau = a$. The torsion of the curves Γ is therefore an absolute constant. The geometric interpretation of (30) is given by (25) or by theorem 4. We have therefore

THEOREM 5. *The circles of a congruence $C_{R\Phi}$ lie in the osculating planes of the curves Γ, all of which are of the same constant torsion and have the product of the distances to the focal points of their congruence of binormals constant.*

From (20) is readily proved

THEOREM 6. *The necessary and sufficient condition that the circles of a congruence C_Φ lie in the osculating planes of the curves Γ is that R be constant along each of their orthogonal trajectories.*

To show the existence of surfaces S with which are associated congruences $C_{R\Phi}$, it is necessary to establish the existence of functions satisfying equations (2), (28), (29) and (30). To this end eliminate p, r, q_1 and r_1 by means of (28), (29), (30) and the last of (2). The resulting equations can be combined so as to give an expression for q in which the only derivatives are ψ_u and ψ_v. When q is eliminated by this relation and the substitutions

$$u_1 = u + v, \qquad v_1 = u - v \tag{33}$$

are made, there results a system which can be solved for ξ_{u_1}, η_{1u_1}, p_{u_1}, and $\psi_{u_1u_1}$. Moreover the expressions for these derivatives are of the form to which Cauchy's existence theorem* can be applied. Hence surfaces S exist. They do not degenerate into curves since the linear element is in the orthogonal form.

If we further impose the restriction that ψ be constant, introduce the new variables

$$\alpha = 1/\varrho, \qquad \beta = 1/\varrho_1, \qquad \gamma = 1/\tau_1,$$

substitute from (3) in (2), (28), (29) and (30), and combine the resulting equations, we find that the conditions for the existence of the corresponding congruence are

$$\xi_v = -\xi\eta_1\alpha\sin\psi, \tag{34}$$

$$\eta_{1u} = c\xi\eta_1, \tag{35}$$

$$\beta\sin\bar{\omega}_1 = c, \tag{36}$$

$$\bar{\omega}_{1v} = \eta_1(\gamma + a), \tag{37}$$

* Jordan, loc. cit.

$$\xi(a+b)\alpha\sin\psi - \xi c\beta\cos\bar{\omega}_1 - (\beta\cos\bar{\omega}_1)_u = 0, \tag{38}$$

$$\alpha_v\cos\psi = \eta_1(\alpha^2\sin\psi\cos\psi - 2ac - \alpha\beta\sin\psi\cos\bar{\omega}_1), \tag{39}$$

$$\alpha_v\sin\psi = \eta_1(\alpha^2\sin^2\psi + c^2 - a^2 + \alpha\beta\cos\psi\cos\bar{\omega}_1), \tag{40}$$

c being a constant defined by the equation

$$c = (b-a)\tan\psi. \tag{41}$$

From (36) follows*

$$(\beta\cos\bar{\omega}_1)_u = -c\csc^2\bar{\omega}_1\,\bar{\omega}_{1u};$$

and from (38)

$$\bar{\omega}_{1u} = \xi c\sin\bar{\omega}_1\cos\bar{\omega}_1 - (a+b)\xi\alpha\sin\psi\sin^2\bar{\omega}_1/c. \tag{42}$$

Multiplying (39) by $\sin\psi$, (40) by $\cos\psi$ and subtracting, we get

$$\alpha\beta\cos\bar{\omega}_1 = -2ac\sin\psi + (a^2-c^2)\cos\psi = k, \tag{43}$$

where k is constant. Thus we have

$$\alpha = k\tan\bar{\omega}_1/c, \tag{44}$$

by the use of which equation (42) becomes

$$\bar{\omega}_{1u} = \xi c\sin\bar{\omega}_1\cos\bar{\omega}_1 - k(a+b)\xi\sin\psi\sin^2\bar{\omega}_1\tan\bar{\omega}_1/c^2. \tag{45}$$

If we differentiate (44) and use (39) or (40), we get

$$\bar{\omega}_{1v} = \eta_1(k\sin^2\bar{\omega}_1\sin\psi/c - 2ac^2\cos^2\bar{\omega}_1/k\cos\psi - c\tan\psi\cos\bar{\omega}_1). \tag{46}$$

Expressing the condition

$$\bar{\omega}_{1uv} = \bar{\omega}_{1vu},$$

and using (34), (35), (44), (45) and (46), we find

$$\xi\eta_1\cos^3\Phi\sin^2\psi\sin^2\bar{\omega}_1 = 0. \tag{47}$$

If we have $\xi\eta_1 = 0$, it is seen from (34) and (35) that the linear element of S assumes one of the forms

$$ds^2 = Udu^2 \qquad \text{or} \qquad ds^2 = Vdv^2,$$

* If $\sin\bar{\omega}_1 = 0$, then from (36) $c = 0$ and either $a = b$, in which case $\Phi = 90°$ and S and its transforms become a plane; or $\psi = 0$. We exclude $c = 0$ for the present.

where U and V are functions of u and v respectively, and that S is therefore a curve. The congruence C degenerates into a single parameter family of circles.

From (47) it is therefore seen that the only possible value for ψ is zero.* We have therefore proved

THEOREM 7. *In a congruence $C_{R\Phi}$ the only possible constant value for ψ is zero.*

Since (36) states that the curves $u =$ const. all have the same constant geodesic curvature, we can express the above result as follows: there exists no curved surface which is the locus of a single parameter family of curves all of which have the same constant torsion, all of whose principal normals make the same constant oblique angle with the corresponding normals to the surface, and whose orthogonal trajectories all have the same constant geodesic curvature.

With $\psi = 0$ and R and Φ constant the fundamental equations become

$$r = 0, \tag{48}$$

$$Rq_1 - \eta_1 \sin\Phi\cos\Phi = 0, \tag{49}$$

$$r_1 p = 0. \tag{50}$$

From (48) the curves Γ are geodesics. If from (50) we choose $r_1 = 0$, S has an orthogonal system of geodesics and is therefore developable. The fundamental quantities for S are then

$$\xi = U, \quad \eta_1 = V, \quad \xi_1 = \eta = r = r_1 = 0,$$
$$p = -aU, \quad p_1 = akV, \quad q = -aU/k, \quad q_1 = aV,$$

where U and V are arbitrary functions of u and v respectively, k is an arbitrary constant, and a is defined by (31).

Choosing $p = 0$, we get from the last equation in (2) $q_1 = 0$. The curves Γ are therefore lines of curvature and, being also geodesics, are plane. S is a surface of Monge.† Finally (49) forces us to choose $\Phi = 0$. The system is therefore cyclic.

THEOREM 8. *If in a congruence $C_{R\Phi}$ whose surface of centers S is not developable the plane of each circle passes through the corresponding normal to S, the circles form a cyclic system and S is a surface of Monge on which the curves Γ are plane geodesics.*

* See preceding foot-note.

† Eisenhart, p. 306.

A particular example of this last type of congruence is furnished by the pseudospherical surface of Monge for which the fundamental quantities are

$$\xi_1 = \eta = p = q_1 = 0, \qquad \xi = a\cos\omega, \qquad \eta_1 = a\sin\omega,$$
$$p_1 = \cos\omega, \qquad q = \sin\omega, \qquad r = \omega_v = 0, \qquad r_1 = \omega_u = \sin\omega.$$

The normal to S_1 is the tangent to the circle at P. Analysis similar to that employed above in obtaining the total curvature of the surfaces S_1 leads to the following values for w_1 and w_2, the principal radii of curvature of S_1,

$$w_1 = -R\cot\theta, \qquad w_2 = -R\tan(\omega-\theta) - a\sin\omega\sec(\omega-\theta).$$

It is easy to give geometric constructions for the principal centers of curvature: the normal to S_1 meets the z axis in the first and the line

$$y = 0, \qquad z = (x+a)\tan\omega \tag{51}$$

in the second. Hence for a given point the principal centers of curvature for all the transformed surfaces S_1 lie on a pair of lines making an angle $90^\circ - \omega$ with each other. This line (51) is the characteristic of the plane of the circle, and the point of contact of this plane with its envelope is the intersection of (51) with the x axis. If from the intersection of (51) with the z axis, we draw tangents to the circle, we get the normals to two transformed surfaces which have the points of contact for umbilical points.

5. S **is a curve.** If the surface of centers reduces to a curve, we can take for the moving triedral the tangent, principal normal and binormal of the curve and for parameter u its arc. With this choice, the translations and rotations have the following values:

$$\xi = 1, \quad \xi_1 = \eta = \eta_1 = q = p_1 = q_1 = r_1 = 0, \quad p = -1/r, \quad r = 1/\varrho. \tag{52}$$

From the fundamental equations for a congruence C we see as before that in general we can choose a relation between R, Φ, ψ and $\varkappa$, or any one of them, without restricting the curve of centers. With $\Phi = 0$, however, the equations become

$$R_v\cos\varkappa + R\varkappa_v\cos^2\psi\sin\varkappa = 0, \tag{53}$$

$$\sin\varkappa(R_v\sin\psi - R\psi_v\cos\psi) = 0, \tag{54}$$

$$r\psi_v + \psi_v\varkappa_u - \psi_u\varkappa_v + p\varkappa_v\cos\varkappa - p\psi_v\tan\psi\sin\varkappa = 0. \tag{55}$$

One set of solutions of these equations is $\sin\psi = \cos\varkappa = 0$. The circles are then in the normal planes to the curve of centers. R is unrestricted.

If $\sin\varkappa$ equals zero, it follows from (53) and (55) either that we have $R_v = \psi_v = 0$, in which case there is only a single infinity of circles; or that r is zero, in which case the curve of centers is a straight line. If we have $\varkappa_v = 0$, $\sin\varkappa \neq 0$ and $\cos\varkappa \neq 0$, (53) and (54) show that there is only a single infinity of circles.

If $\varkappa_v$ and r do not vanish, then ψ is zero. For assuming $\psi \neq 0$, we get from (53)

$$\tan\psi = U\cos\varkappa. \tag{56}$$

Substitution in (55) gives

$$\varkappa_v\left[(p + pU^2 - U')\cos\varkappa - rU\sin\varkappa\right] = 0.$$

This gives under the assumptions $U = 0$ and hence from (56), $\psi = 0$. Equations (53), (54) and (55) can be replaced for $\psi = 0$ by

$$R = U\cos\varkappa, \tag{57}$$

$$p = 0. \tag{58}$$

From (58) and (52), it follows that the curve of centers is plane.

We can state therefore

THEOREM 9. *If the centers of the circles of a cyclic system lie on a curve, then*

(i) *The circles lie in the normal planes to the curve of centers and their radii are given by an arbitrary function; or*

(ii) *The center locus is a straight line; or*

(iii) *The curve of centers is plane and the circles lie in planes through the binormals.*

A cyclic system with a given plane curve for locus of centers can be constructed in the following way. Choose an arbitrary function U of the arc. At each point of the curve construct the circles

$$x^2 + y^2 = \pm Ux,\ z = 0. \tag{59}$$

Any circle which lies in a plane perpendicular to the plane of the given curve and which has for diameter a line segment passing through the origin and terminating in the two circles (59) is a circle of the congruence.

When combined with (52), the fundamental equations (20), (21) and (22) become, $\varkappa$ being zero,

$$R^2(\psi_v \Phi_u - \psi_u \Phi_v) - R_v \cos^2\Phi + R^2 p \Phi_v = 0, \tag{60}$$

$$r \Phi_v \cos\psi = 0, \tag{61}$$

$$r \psi_v \cos\psi = 0. \tag{62}$$

The assumption $r = 0$ leads to a straight line. The choice $\Phi_v = \psi_v = 0$ gives $R_v = 0$ from (60). Hence if there is a double infinity of circles and the center locus is not a straight line, ψ is 90° and the circles lie in the osculating planes. Equation (60) when integrated for this value of ψ becomes

$$1/R + p \tan\Phi = U, \tag{63}$$

where U is an arbitrary function of u.

6. S is a point. The trihedral for the surface is indeterminate, so ψ can be taken equal to 90°. Moreover, the translations all vanish. Hence the fundamental equations are

$$p \Phi_v - p_1 \Phi_u = 0, \qquad q \Phi_v - q_1 \Phi_u = 0, \qquad p q_1 - p_1 q = 0.$$

The last equation states that the circles lie in a single infinity of planes. The above system can be replaced by

$$\Phi_u = e^f p, \tag{64}$$

$$\Phi_v = e^f p_1, \tag{65}$$

$$p q_1 - p_1 q = 0. \tag{66}$$

The condition $\Phi_{uv} = \Phi_{vu}$ gives

$$f_u = (q r_1 - q_1 r + p f_v)/p_1. \tag{67}$$

Equation (66) is satisfied by the rotations for any developable surface. With such a choice of p, p_1, q and q_1 there exists a solution of (67) which when substituted in (64) and (65) gives Φ. There is no restriction on the radius. If Φ is constant, the only condition is (66).

UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PENNSYLVANIA.

ON NEUMANN'S EXISTENCE PROOF.

By William F. Osgood.

In his proof of the existence theorem for the Abelian integrals connected with a given algebraic manifold, Neumann* requires one lemma, the proof of which, although based on a simple, geometric interpretation of a line integral, nevertheless demands some ingenuity. It is the purpose of this note to give a direct proof by means of the methods of analysis which, since that time, have become the common property of mathematicians.

Let a function f be continuous along the circumference of a circle C, and let its maximum and minimum values on C be denoted respectively by G and K:

$$K \leq f \leq G,$$

each lower sign holding in at least one point of C. Then the difference,

$$\Delta = G - K,$$

shall be called the *oscillation* of f on C, and shall be denoted as follows:

$$Df\big|_C = \Delta.$$

The Lemma. *Let u be harmonic throughout the interior of a circle C and continuous on the boundary. Let Γ be a concentric circle of smaller radius. Then there exists a positive constant q less than unity, depending on the choice of Γ, but not on the choice of u, such that*

$$Du\big|_\Gamma \leqq q\, Du\big|_C. \tag{1}$$

Suppose the theorem false. Then there would exist a sequence of functions $u_1, u_2, \ldots$, each harmonic in C and continuous on the circumference, and such that

$$Du_n\big|_\Gamma > q_n\, Du_n\big|_C, \tag{2}$$

where

$$q_n < q_{n+1}, \qquad \lim_{n=\infty} q_n = 1.$$

* Abelsche Integrale, 1884, p. 417.

Moreover, these functions may obviously be so taken as to have all the same maximum value G and the same minimum value K on C:

$$K \leqq u_n \leqq G. \qquad \text{Thus} \qquad Du_n|_C = G - K.$$

Denote the maximum and the minimum values of u_n on Γ respectively by

$$\mathfrak{M}u_n|_\Gamma, \qquad \mathfrak{m}u_n|_\Gamma.$$

Then (2) becomes:

$$\mathfrak{M}u_n|_\Gamma - \mathfrak{m}u_n|_\Gamma > q_n G - q_n K. \tag{3}$$

Since $\mathfrak{M}u_n|_\Gamma$ and $\mathfrak{m}u_n|_\Gamma$ both lie between G and K, it follows at once from (3) that:

$$G - \mathfrak{m}u_n|_\Gamma > q_n G - q_n K,$$

$$\mathfrak{m}u_n|_\Gamma < q_n K + (1 - q_n)G = K + (1 - q_n)(G - K). \tag{4}$$

Similarly,

$$G - (1 - q_n)(G - K) < \mathfrak{M}u_n|_\Gamma. \tag{5}$$

Let the point A be chosen on Γ and held fast. It is clear that u_n may be so taken that it attains the value $\mathfrak{M}u_n|_\Gamma$ in the point A. Thus

$$\lim_{n=\infty} u_n = G \qquad \text{in } A.$$

On the other hand, let S be any closed region lying within C and containing Γ in its interior,—for example, the interior and circumference of a circle A concentric with C and Γ and lying between them. Then we can select from the sequence $u_1, u_2, \ldots$ a sub-set,

$$u_{m_1}, u_{m_2}, \ldots,$$

which will converge uniformly throughout S and thus define a function u:

$$u = \lim_{n=\infty} u_{m_n}.$$

harmonic within S and continuous on the boundary.* Moreover, since

$$K < u_n \Big|_A < G,$$

it follows that

$$K \leq u \Big|_A \leq G.$$

But u attains the value G in A. — an interior point of A. Hence u is constant throughout A.

$$u = G.$$

This result contradicts the fact that u_{m_n} converges uniformly along Γ and, because of (4), falls below the value

$$K + (1 - q_{m_n})(G - K)$$

at some point of Γ. Hence the proof.

It may be remarked that the foregoing proof also establishes the lemma for the region consisting of an arbitrary algebraic Riemann's surface, from which the interior of a circle C has been removed, provided C contains at most one branch-point. Γ is then any closed curve lying in the surface and not meeting C.

* For the proof of this theorem and the citation of the literature cf. the author's paper: "On the Uniformization of Algebraic Functions", these Annals (2) 14 (1913), p. 152.

HARVARD UNIVERSITY.

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ON THE AUTOMORPHIC FUNCTIONS OF THE GROUP (0, 3; 2, 4, 6).

By H. H. Dalaker.

Introduction. The infinite discontinuous groups of sign $(0, 3; l_1, l_2, l_3)$* have been discussed by Morris in his dissertation.† He there shows that, of the possible groups of this character whose uniform automorphic functions are expressible in theta series, the group (0, 3; 2, 4, 6) is the only one which has not been studied in detail. The associated Riemann surface

$$w^{24} = (z-a)(z-b)^{11}(z-c)^{17}(z-d)^{19}$$

is of genus 23. The methods employed in the detailed study of the other groups of this character are too cumbersome to be available for this group.

Accordingly, following a method suggested by Hilbert, a simple table of hyperelliptic moduli is assumed and its group is investigated. This group is of sign (0, 3; 2, 4, 6); its functions and those of some of its subgroups are uniformized. No attempt has been made to show that the group here studied coincides with that of the above surface, but it is hoped that this investigation may throw light upon the general group.

This paper consists of five parts. In parts *one* and *two* the arithmetic character of the group, and some of its subgroups are discussed. In the *third* part the Poincaré series for the group and one of its subgroups are determined. Part *four* deals with the theta transformations and the uniformization of the functions of the main group. In the last part the uniformization of the functions belonging to some of the subgroups is discussed.

I wish to acknowledge my indebtedness to Prof. J. I. Hutchinson, who suggested the problem and directed the earlier part of the work, and to Prof. C. F. Craig, who assisted in its completion.

§ 1. **The group (0, 3; 2, 4, 6).** Following a method suggested by Hilbert‡ we assume the table

* See Fricke u. Klein, Automorphe Functionen, vol. 1, p. 353, for this notation. This work will be referred to as Fricke-Klein.

† On the automorphic functions of the group $(0, 3; l_1, l_2, l_3)$, Trans. Amer. Math. Soc., vol. 7, 1906, pp. 425–448. In his introduction there are sufficient indications of the bibliography to suggest what has been done.

‡ See O. Blumenthal, Jahresb. der Deutschen Math.-Ver., vol. 13, 1904, pp. 121–123.

(1)

	a_1	a_2	b_1	b_2
u_1	$\sqrt{2}\,\lambda_2$	λ_2	$\sqrt{2}\,\lambda_1$	$2\lambda_1$
u_2	$-\sqrt{2}\,\mu_2$	μ_2	$-\sqrt{2}\,\mu_1$	$2\mu_1$

as a table of moduli of a pair of hyperelliptic integrals u_1, u_2 of the first kind. Reducing the above to the canonic form, the Weierstrass moduli are

$$\tau_{11} = \frac{\lambda+\mu}{2}, \qquad \tau_{12} = \tau_{21} = \frac{\lambda-\mu}{2}, \qquad \tau_{22} = \lambda+\mu,$$

in which $\lambda = \lambda_1/\lambda_2$, $\mu = \mu_1/\mu_2$ and λ_1, λ_2, μ_1, μ_2 are independent variables. It will be observed that the moduli satisfy

(2) $$2\tau_{11} - \tau_{22} = 0.$$

We now assume that they satisfy the further relation

(2′) $$2\tau_{11} + (\tau_{12}\tau_{21} - \tau_{11}\tau_{22}) - 2 = 0,$$

thus making the corresponding hyperelliptic theta functions doubly singular.* In order that (2′) shall be satisfied we must have

(3) $$\lambda + \mu - 2\lambda\mu - 2 = 0.$$

This relation may be expressed in the form

$$\frac{\lambda-\varepsilon}{\lambda+\varepsilon^2} + \frac{\mu-\varepsilon}{\mu+\varepsilon^2} = 0, \qquad \left(\varepsilon = \frac{1+\sqrt{-3}}{2}\right).$$

Hence, writing

$$\lambda_1 = \varepsilon^2\eta_1 + \varepsilon\eta_2, \quad \lambda_2 = -\eta_1 + \eta_2, \quad \mu_1 = -\varepsilon^2\eta_1 + \varepsilon\eta_2, \quad \mu_2 = \eta_1 + \eta_2,$$

table (1) becomes

(4)

	a_1	a_2	b_1	b_2
u_1	$\sqrt{2}\,(-\eta_1+\eta_2)$	$-\eta_1+\eta_2$	$\sqrt{2}\,(\varepsilon^2\eta_1+\varepsilon\eta_2)$	$2(\varepsilon^2\eta_1+\varepsilon\eta_2)$
u_2	$-\sqrt{2}\,(\eta_1+\eta_2)$	$\eta_1+\eta_2$	$\sqrt{2}\,(\varepsilon^2\eta_1-\varepsilon\eta_2)$	$-2(\varepsilon^2\eta_1-\varepsilon\eta_2)$

in which η_1, η_2 are independent variables.

* See G. Humbert, Liouville, Ser. 5, vol. 9, 1903, p. 65 and following.

The Weierstrass moduli which satisfy (2) and (2′) are now

$$\tau_{11} = \frac{\varepsilon^2\eta^2+\varepsilon}{1-\eta^2}, \qquad \tau_{12} = \tau_{21} = \frac{\sqrt{2}\,(\varepsilon+\varepsilon^2)\eta}{1-\eta^2}, \qquad \tau_{22} = 2\tau_{11},$$

where $\eta = \eta_1/\eta_2$. The condition for the convergence of the corresponding theta series takes the particularly simple form $1-\eta\bar{\eta} > 0$.

We next apply the transformations

$$\eta_1' = a\eta_1 + b\eta_2, \qquad \eta_2' = c\eta_1 + d\eta_2 \tag{5}$$

to the variables η_1, η_2 in table (4). Furthermore we assume that the same result is obtained by applying the transformation of integers

$$\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \delta \end{array}$$

to (4) or to (4) with the signs of all the elements in the last row changed. This leads to the two types of transformations on* η.

$$(\alpha)\ \begin{vmatrix} a_1-2\varepsilon a_2 & \sqrt{2}\,(b_1-\varepsilon b_2) \\ \sqrt{2}\,(b_1+\varepsilon^2 b_2) & a_1+2\varepsilon^2 a_2 \end{vmatrix}, \qquad (\beta)\ \begin{vmatrix} \sqrt{2}\,(b_1-\varepsilon b_2) & a_1-2\varepsilon a_2 \\ a_1+2\varepsilon^2 a_2 & \sqrt{2}\,(b_1+\varepsilon^2 b_2) \end{vmatrix}$$

where a_1, a_2, b_1, b_2 are integers satisfying

$$a_1^2 - 2a_1a_2 + 4a_2^2 - 2(b_1^2 - b_1b_2 + b_2^2) = 1.$$

The corresponding transformations of integers are

$$(\alpha')\ \begin{array}{cc|c} a_1-2a_2 & -2b_1 & 2a_2\ b_2 \\ -b_1 & a_1-2a_2 & b_2\ \ a_2 \\ \hline -2a_2 & -2b_1+2b_2 & a_1\ \ b_1 \\ -2b_1+2b_2 & -4a_2 & 2b_1\ \ a_1 \end{array}, \qquad (\beta')\ \begin{array}{cc|c} -a_1 & 2b_1-2b_2 & 2a_2\ b_2 \\ b_1-b_2 & -a_1 & b_2\ \ a_2 \\ \hline -a_1+2a_2 & -2b_2 & a_1\ \ b_1 \\ -2b_2 & -2a_1+4b_2 & b_1\ \ a_1 \end{array}.$$

* In conformity with usage $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ stands for $\eta' = \dfrac{a\eta+b}{c\eta+d}$.

From the relation

$$\sum_{x=1}^{2} \tau'_{\nu x}\left(\alpha_{x\mu} + \sum_{\lambda=1}^{2} \beta_{x\lambda}\, \tau_{\mu\lambda}\right) = \gamma_{\nu\mu} + \sum_{\lambda=1}^{2} \delta_{\nu\lambda}\, \tau_{\mu\lambda},$$

where $\tau'_{\nu x}$ is the transform of $\tau_{\nu x}$ under (5), other transformations of the group may be determined. It is found that the transformation

$$S: \begin{vmatrix} \varepsilon^2 & 0 \\ 0 & \varepsilon \end{vmatrix}$$

belongs to the group. Its transformation of integers is

$$\left|\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right|.$$

By applying S^n $(n = 1, 2, \ldots, 6)$ to (α) and (β) there result twelve classes of transformations which reduce to the four distinct types

I

$$\begin{vmatrix} \alpha + \beta\varepsilon & \sqrt{2}\,(\gamma + \delta\varepsilon) \\ \sqrt{2}\,(\gamma - \delta\varepsilon^2) & \alpha - \beta\varepsilon^2 \end{vmatrix},$$

II

$$\begin{vmatrix} \sqrt{2}\,(\alpha + \beta\varepsilon) & \gamma + \delta\varepsilon \\ \gamma - \delta\varepsilon^2 & \sqrt{2}\,(\alpha - \beta\varepsilon^2) \end{vmatrix},$$

III

$$\begin{vmatrix} \alpha + \beta\varepsilon & \sqrt{2}\,(\gamma + \delta\varepsilon) \\ -\sqrt{2}\,(\gamma - \delta\varepsilon^2) & -(\alpha - \beta\varepsilon^2) \end{vmatrix},$$

IV

$$\begin{vmatrix} \sqrt{2}\,(\alpha + \beta\varepsilon) & \gamma + \delta\varepsilon \\ -(\gamma - \delta\varepsilon^2) & -\sqrt{2}\,(\alpha - \beta\varepsilon^2) \end{vmatrix},$$

where $\alpha, \beta, \gamma, \delta$ are integers such that for I and III

$$\alpha^2 + \alpha\beta + \beta^2 - 2(\gamma^2 + \gamma\delta + \delta^2) = 1, \tag{6}$$

and for II and IV

$$2(\alpha^2 + \alpha\beta + \beta^2) - (\gamma^2 + \gamma\delta + \delta^2) = 1.$$

That the totality of transformations of the above types forms a group having for its fixed circle

$$1 - \eta\bar{\eta} = 0 \qquad (\bar{\eta} = x - iy)$$

is readily verified. These four types of transformations define the group which we will call G.

Determination of the fundamental region. Following the method of Hutchinson* we observe that for the circles†

$$(c\eta + d)\ (*) = 1$$

nearest to the center of $\eta\bar{\eta} - 1 = 0$ and enclosing a region the expression $\lambda\bar{\lambda} = d\bar{d}$ must be a minimum. For the types I and III $\lambda\bar{\lambda} = \alpha^2 + \alpha\beta + \beta^2$, for II and IV $\lambda\bar{\lambda} = 2(\alpha^2 + \alpha\beta + \beta^2)$, whence in all cases $\alpha^2 + \alpha\beta + \beta^2$ must be as small as possible.

It is found that the boundary circles cutting nearest the center of the fixed circle are obtained from either of the types II, IV. Taking the least possible value for $\lambda\bar{\lambda}$,

$$\alpha^2 + \alpha\beta + \beta^2 = 1,$$

whence

$$\gamma^2 + \gamma\delta + \delta^2 = 1$$

there result the six circles, and no others,

$$(8) \qquad |\eta - \theta\sqrt{2}| \cdot [*] = 1, \qquad \theta = \pm 1,\ \pm\epsilon,\ \pm\epsilon^2,$$

which enclose a region R whose vertices are designated in Fig. 1 by $A_1, A_2, \ldots, A_{18}$.

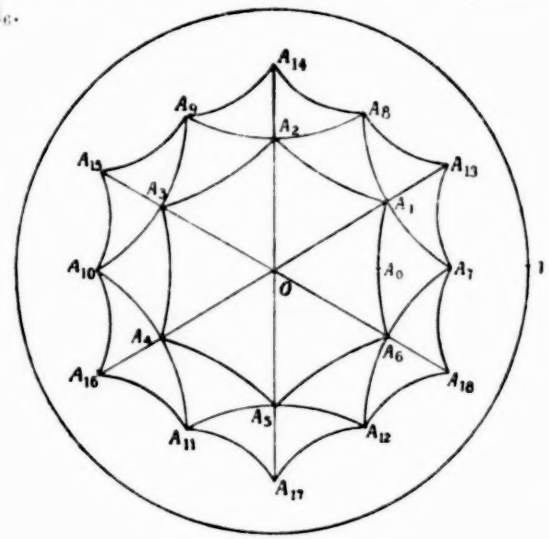

Fig. 1.

* See Trans. Amer. Math. Soc., vol. 8, 1907, pp. 261–269.

† The notation is $(*) = (\bar{c}\,\bar{\eta} + \bar{d})$.

If larger values of $\lambda\bar{\lambda}$ are taken, or if the same process is applied to types I, II, other circles are found, but none of these cut inside of R; at most they pass only through its vertices.

The transformation

$$T: \begin{vmatrix} \sqrt{2}- & 1 \\ 1 & -\sqrt{2} \end{vmatrix},$$

belongs to the group. Moreover T pairs $A_0 A_6$ with $A_0 A_1$, and S pairs $0A_6$ with $0A_1$; it follows that we may take the quadrilateral $0A_6 A_0 A_1$ as the fundamental region R. The cycles of vertices are $q = 3$, the number of sides $2n = 4$, whence the genus p is 0 since* $p = (1+n-q)/2$.

The transformation leaving the vertex A_1 unaltered, pairing the arc $A_6 A_1$ with the arc $A_2 A_1$ is

$$(ST)^{-1} = U: \begin{vmatrix} \sqrt{2}\,\varepsilon & -\varepsilon^2 \\ \varepsilon & -\sqrt{2}\,\varepsilon^2 \end{vmatrix}.$$

Since $T^2 = 1$, $U^4 = 1$, $S^6 = 1$, we have a group of genus 0 with *three* cycles of vertices for R and transformations of periods 2, 4, 6, leaving respectively the noncongruent vertices fixed. Hence the symbol (0, 3; 2, 4, 6) characterizes the group and may be used to represent it.

Periods of the transformations of G. A transformation $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is of period n provided† $\dfrac{(a+d)^2}{ad-bc} = 4\cos^2\dfrac{\pi l}{n}$ where l is an integer and $0 \leqq l < n$. Applying this criterion we find that the transformations are hyperbolic or elliptic of periods 2, 3, 4 and 6. There are no parabolic transformations.

§ 2. **Subgroups of (0, 3; 2, 4, 6).** In the investigation of the subgroups of G those whose genus is less than *two* have in general been discarded since illustrations of their occurrence in the problem of uniformization are fairly frequent. Thus those obtained by pairing the sides of the hexagon $A_1, A_2, \ldots, A_6$ have all been investigated and found to be of genus *zero* or *one*. Many others have been examined: the following have been retained as of special interest. Apparently the genus may be extended beyond the limit here found.

The subgroup (0, 3; 2, 6, 6). The totality of transformations of types I and III form a subgroup G_1, whose boundary circles are

$$[\sqrt{2}\,\eta - \theta(1+\varepsilon)]\,[^*] = 1, \qquad \theta = \pm 1, \pm\varepsilon, \pm\varepsilon^2.$$

* See Forsyth, A. R., Theory of functions, § 293.

† See Forsyth, l. c., § 258.

These enclose the region $A_7, A_8, \ldots A_{12}, A_7$ and pass through the vertices $A_1, A_2, \ldots A_6$. The transformations S and

$$(TS^{-1})^2 = T_1: \begin{vmatrix} \varepsilon + \varepsilon^2 & -\sqrt{2}\,\varepsilon^2 \\ \sqrt{2}\,\varepsilon & -(\varepsilon + \varepsilon^2) \end{vmatrix}$$

belong to this group; S pairs $0A_7$ with $0A_8$, while T_1 pairs A_1A_7 with A_1A_8. Furthermore $U_1 = S^{-1}T_1$ leaves A_7 invariant, and hence $0A_7A_1A_8$ is the fundamental region, the group sign being $(0, 3; 2, 6, 6)$.

The subgroup (α). It will be observed that the transformations of type (α), § 1 form a subgroup of G, whose fundamental region is $A_7A_8 \cdots A_{12}A_7$. The generating transformations are

$$T_1, \quad ST_1S^{-1}, \quad S^2T_1S^{-2}, \quad \ldots \quad S^5T_1S^{-5};$$

they are all of period *two*, and leave the points $A_1, A_2, \ldots A_6$ invariant. Each circle is paired with itself. The group sign is $(0, 7; 2, 2, 2, 2, 2, 2)$.

Subgroups of genus 2. The transformations

$$\begin{vmatrix} 1 + A(1-\varepsilon^2) & \sqrt{2}\,B(1-\varepsilon^2) \\ \sqrt{2}\,B(1+\varepsilon) & 1 + \bar{A}(1+\varepsilon) \end{vmatrix} \text{ and } \begin{vmatrix} 1 + A(1-\varepsilon^2) & \sqrt{2}\,B(1-\varepsilon^2) \\ -\sqrt{2}\,B(1+\varepsilon) & -1 - \bar{A}(1+\varepsilon) \end{vmatrix}$$

form a subgroup of G. Here $A = a_1 + \varepsilon^2 a_2$, $B = b_1 + \varepsilon^2 b_2$ (a_1, a_2, b_1, b_2 integers) satisfy

$$[1 + A(1-\varepsilon^2)][*] - 6B\bar{B} = 1.$$

The boundary circles are

$$[\sqrt{2}(1+\varepsilon)\eta - \theta(2-\varepsilon^2)][*] = 1, \quad [\sqrt{2}(1+\varepsilon)\eta - \theta(2+\varepsilon)][*] = 1,$$
$$\theta = \pm 1, \pm\varepsilon, \pm\varepsilon^2.$$

In the figure these are A_7A_{13}, $A_{13}A_8$, etc. The total group may be generated by S and $(STS)^2$.

All the subgroups belonging to this new polygon have been investigated. It is found that there are *eight* of genus *two* and none of higher genus. One of these is the subgroup g_1. The sides are paired as follows

$$T_2 = ST_1S^{-1}T_1 \text{ pairs } A_7A_{13} \text{ with } A_9A_{14},$$
$$ST_2S^{-1} \quad \text{,,} \quad A_8A_{14} \quad \text{,,} \quad A_{10}A_{15}.$$

while $S^2 T_2 S^{-2}$, $S^3 T_2 S^3$, $S^{-2} T_2 S^2$, $S^{-1} T_2 S$ pair the remaining ones. The other seven subgroups of genus 2 obtained from the same polygon by different methods of pairing the sides are identical with g_1 since the generating transformations are *hyperbolic* in each case.*

Subgroup of genus 5. The transformations

$$\begin{vmatrix} 1+A(2-\varepsilon^2) & \sqrt{2}\,B(2-\varepsilon^2) \\ \sqrt{2}\,\bar{B}(2+\varepsilon) & 1+\bar{A}(2+\varepsilon) \end{vmatrix} \text{ and } \begin{vmatrix} 1+A(2-\varepsilon^2) & \sqrt{2}\,B(2-\varepsilon^2) \\ -\sqrt{2}\,\bar{B}(2+\varepsilon) & -1-\bar{A}(2+\varepsilon) \end{vmatrix}$$

form a subgroup of G. Here A and B are defined as above and satisfy the relation

$$[1-A(2-\varepsilon^2)][*]-14\,B\bar{B}=1.$$

The circles

$$[\sqrt{2}(1+4\varepsilon)\eta+\theta(1-7\varepsilon^2)][*]=1, \quad [\sqrt{2}(1+4\varepsilon)\eta+\theta(7\varepsilon-6)][*]=1,$$
$$[2\sqrt{2}(2+\varepsilon)\eta+\theta(8-7\varepsilon)][*]=1, \quad [2\sqrt{2}(2+\varepsilon)\eta+\theta(1+7\varepsilon)][*]=1,$$

where $\theta=\pm 1, \pm\varepsilon, \pm\varepsilon^2$ are the exterior boundary of the fundamental region. Consider now the pairing of the sides as was done in the preceding case. Since the circles are not all of the same radii, there is necessarily less freedom of choice in the modes of pairing. One such is (see Fig. 2):

$$r_1 = STS^3\,TS^5\,TST. \quad \text{pairs } A_{19}A_{20} \text{ with } A_{24}A_{23},$$
$$r_2 = STS^5\,TS^3\,TS^4\,TS^5, \quad \text{“} \quad A_{20}A_{21} \quad \text{“} \quad A_{25}A_{24}.$$

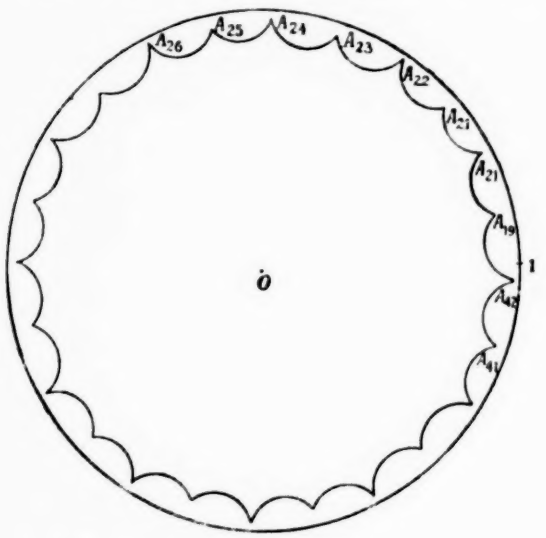

Fig. 2.

* See Fricke-Klein, vol. 1, p. 261 and following.

while

$$S_\nu = S^\nu \tau_r S^{-\nu}, \qquad r = 1, 2; \qquad \nu = 1, 2, \ldots, 5$$

pair the remaining sides. There are 3 cycles of vertices and since $n = 12$, the genus is 5.

§ 3. **Poincaré series for G and G_1.** In order to determine the automorphic functions belonging to the group, it will first be necessary to construct certain Poincaré series.*

A function of the form

$$P(\eta) = \sum_{i=1}^{\infty} \frac{H(\eta_i)}{\mu_i (c_i \eta + d_i)^m}, \qquad \eta_i = \frac{a_i \eta + b_i}{c_i \eta + d_i}, \qquad a_i d_i - b_i c_i = 1,$$

where $H(\eta_i)$ is a rational function of η_i and the summation is extended to all the substitutions of the group, is a generalized Poincaré series of degree m. The series $P(\eta)$ converges absolutely for $m > 2$, and satisfies the relation

$$P(\eta_i) = \mu_i (c_i \eta + d_i)^m P(\eta). \tag{1}$$

At the vertices A_i, the fixed points of elliptic substitutions of period l_i of a fundamental region R, every function $P(\eta)$ of degree m has certain fixed zeros in number ν_i. In addition to these there are simple zeros of $P(\eta)$ inside of R, or on its boundary, which change position when the constants in $H(\eta)$ are altered. These *movable* zeros are N in number where†

$$N = \tfrac{1}{2} m(n-1) - \tfrac{1}{2} \sum_i \frac{2\nu_i + m}{l_i}, \tag{2}$$

$2n$ being the number of sides of R, with the summation extending to all the cycles of vertices.

The number of linearly independent functions $P_i(\eta)$ of the same multipliers μ and of degree m is $N+1$, hence a linear relation exists among $N+2$ such functions. Also $P(\eta) = \sum_1^s A_i P_i(\eta)$, where A_i are constants, will be a Poincaré series of order m and multipliers μ if each $P_i(\eta)$ is such a series. Finally we have‡

$$\mu_i = e^{\pi i (2\nu_i + m)/l_i}. \tag{3}$$

* See, for instance, Hutchinson, On the automorphic functions of the group $(0, 3; 2, 6, 6)$, Trans. Amer. Math. Soc., vol. 5, 1904, pp. 447–460. In particular see § 2.

† Cf. Forsyth, Theory of functions, § 307.

‡ See Hutchinson, l. c., § 2.

Determination of $\psi_1(\eta)$, $\psi_2(\eta)$, $\psi_3(\eta)$. It is necessary to determine the Poincaré series of lowest degree which vanish to the first order at the noncongruent vertices of the fundamental region. For the group G there are three cycles of vertices in R, and accordingly three such series will be constructed, vanishing to the first order respectively at the vertices $A_1, 0, A_0$, and nowhere else in R. Formula (3) requires that the determinant of the substitution be unity. Consequently S, T, U must be written in the unimodular forms

$$(4) \qquad S: \begin{vmatrix} i\varepsilon^2 & 0 \\ 0 & i\varepsilon \end{vmatrix}, \quad T: \begin{vmatrix} \sqrt{2}\, i & -i \\ i & -\sqrt{2}\, i \end{vmatrix}, \quad U: \begin{vmatrix} \sqrt{2}\,\varepsilon & -\varepsilon^2 \\ \varepsilon & -\sqrt{2}\,\varepsilon^2 \end{vmatrix}.$$

From (3) the corresponding multipliers now become

$$\mu_S = e^{\pi i\left(\frac{\nu_S}{3}+\frac{m}{6}\right)}, \qquad \mu_T = e^{\pi i\left(\nu_T+\frac{m}{2}\right)}, \qquad \mu_U = e^{\pi i\left(\frac{\nu_U}{2}+\frac{m}{4}\right)}.$$

Since, however, $(ST)^{-1} = U$, there must exist the relation $(\mu_S\mu_T)^{-1} = \mu_U$. As m can only be even for the group* the simplest cases are

$$\begin{aligned} &(a)\ m=4,\ \ \nu_S=1,\ \ \nu_T=\nu_U=0;\ \ \mu_S=-1,\ \mu_T=1,\ \ \ \mu_U=-1;\\ (5)\ &(b)\ m=6,\ \ \nu_U=1,\ \ \nu_S=\nu_T=0;\ \ \mu_S=-1,\ \mu_T=-1,\ \mu_U=1;\\ &(c)\ m=12,\ \nu_T=1,\ \ \nu_S=\nu_U=0;\ \ \mu_S=1,\ \ \ \mu_T=-1,\ \mu_U=-1. \end{aligned}$$

We denote the functions corresponding to these by $\psi_1(\eta)$, $\psi_2(\eta)$, $\psi_3(\eta)$ respectively. These functions satisfy

$$\begin{aligned} &(a)\ \ \psi_1(\varepsilon\eta) = \varepsilon\,\psi_1(\eta), \quad \psi_1\left(\frac{\sqrt{2}\,i\eta - i}{i\eta - i\sqrt{2}}\right) = \ \ (i\eta - \sqrt{2}\,i)^4\ \psi_1(\eta);\\ (6)\ &(b)\ \ \psi_2(\varepsilon\eta) = \ \ \psi_2(\eta), \quad \psi_2\left(\frac{\sqrt{2}\,i\eta - i}{i\eta - i\sqrt{2}}\right) = -(i\eta - \sqrt{2}\,i)^6\ \psi_2(\eta);\\ &(c)\ \ \psi_3(\varepsilon\eta) = \ \ \psi_3(\eta), \quad \psi_3\left(\frac{\sqrt{2}\,i\eta - i}{i\eta - i\sqrt{2}}\right) = -(i\eta - \sqrt{2}\,i)^{12}\,\psi_3(\eta). \end{aligned}$$

* Hutchinson, l. c., p. 454.

The function $\psi_1(\eta)$ vanishes to the first order at $\eta = 0$, but since S is of period six there are at this fixed point six congruent regions to R sharing this same zero. Now $\psi_1^6(\eta)$ is an absolutely convergent Poincaré series vanishing to the 6th order at $\eta = 0$, and we may say that it vanishes to the *first* order at this point as regards R. Similarly the functions $\psi_2^4(\eta)$, $\psi_3^2(\eta)$ vanish to the *first* order at the fixed points of U and T respectively. The three functions $\psi_1^6(\eta)$, $\psi_2^4(\eta)$, $\psi_3^2(\eta)$ are Poincaré series having the same multipliers $\mu = 1$ and degree $m = 24$. Each has a single movable zero belonging to R at the respective fixed points. For the multipliers are unity, and hence $(2\nu + m)/l$ is an even integer, or which is the same, $\nu + m/2 \equiv 0 \pmod{l}$. We have then $\nu_S + 12 \equiv 0 \pmod{6}$, $\nu_U + 12 \equiv 0 \pmod{4}$, $\nu_T + 12 \equiv 0 \pmod{2}$, whence taking ν_S, ν_U, ν_T the least positive residues, there result $\nu_S = \nu_T = \nu_U = 0$. Substituting these values in formula (2) we get $N = 1$.

To sum up, the three functions $\psi_1^6(\eta)$, $\psi_2^4(\eta)$, $\psi_3^2(\eta)$ are Poincaré series of the same *degree* and *multipliers*. Each has a single movable zero at the fixed points of S, U, T respectively and vanishes nowhere else in R. Hence the three series are linearly dependent. We may therefore write

$$A\,\psi_1^6(\eta) + B\,\psi_2^4(\eta) + C\,\psi_3^2(\eta) = 0, \tag{7}$$

where A, B and C are constants.

It will be the object of the next section to show that these series do not vanish identically. It will then follow that the ratios of the constants can be determined. Any other Poincaré series which belongs to the group can be expressed as a product of $\psi_1(\eta)$, $\psi_2(\eta)$, $\psi_3(\eta)$ and functions of the form $A\,\psi_1^6(\eta) + B\,\psi_2^4(\eta)$.

Determination of $\varphi_1(\eta)$, $\varphi_2(\eta)$, $\varphi_3(\eta)$. The Poincaré series of the subgroup (0, 3; 2, 6, 6) are of special importance in the uniformizing of the functions of the subgroups of genus 2. The simplest cases are*

(a) $m = 2$, $\nu_S = 1$, $\nu_{T_1} = \nu_{U_1} = 0$; $\mu_S = \varepsilon^2$, $\mu_{U_1} = \varepsilon$, $\mu_{T_1} = -1$;

(b) $m = 2$, $\nu_{T_1} = 1$, $\nu_S = \nu_{U_1} = 0$; $\mu_S = \varepsilon$, $\mu_{U_1} = \varepsilon^2$, $\mu_{T_1} = -1$;

(c) $m = 6$, $\nu_{U_1} = 1$, $\nu_S = \nu_{T_1} = 0$; $\mu_S = -1$, $\mu_{U_1} = -1$, $\mu_{T_1} = 1$.

We denote the corresponding functions by $\Phi_1(\eta)$, $\Phi_2(\eta)$, $\Phi_3(\eta)$ respectively. They satisfy the functional relations

* See Hutchinson, l. c., for convergence when $m = 2$.

$$(a)\quad \Phi_1(\varepsilon\eta) = \varepsilon\Phi_1(\eta), \quad \Phi_1\left(\frac{(\varepsilon+\varepsilon^2)\eta - \sqrt{2}\varepsilon^2}{\sqrt{2}\varepsilon\eta - (\varepsilon+\varepsilon^2)}\right) = -[\sqrt{2}\varepsilon\eta - (\varepsilon+\varepsilon^2)]^2\,\Phi_1(\eta);$$

$$(8)(b)\quad \Phi_2(\varepsilon\eta) = \Phi_2(\varepsilon), \quad \Phi_2\left(\frac{(\varepsilon+\varepsilon^2)\eta - \sqrt{2}\varepsilon^2}{\sqrt{2}\varepsilon\eta - (\varepsilon+\varepsilon^2)}\right) = -[\sqrt{2}\varepsilon\eta - (\varepsilon+\varepsilon^2)]^2\,\Phi_2(\eta);$$

$$(c)\quad \Phi_3(\varepsilon\eta) = \Phi_3(\eta), \quad \Phi_3\left(\frac{(\varepsilon+\varepsilon^2)\eta - \sqrt{2}\varepsilon^2}{\sqrt{2}\varepsilon\eta - (\varepsilon+\varepsilon^2)}\right) = [\sqrt{2}\varepsilon\eta - (\varepsilon+\varepsilon^2)]^6\,\Phi_3(\eta).$$

§ 4. The theta transformations. To the transformation S on η there is the corresponding one of integers on the theta moduli

$$\left.\begin{array}{cc|cc} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right..$$

This leads to the relation*

$$(1)\quad \frac{1}{\sqrt{1-\eta'^2}}\,\vartheta\begin{bmatrix} -\frac{1}{2}+g_1+h_1 & g_2 \\ -g_1 & h_2 \end{bmatrix}_{\eta'}(0) = \frac{1}{\sqrt{1-\eta^2}}\,e^{-\pi i(g_1^2+2g_1g_2-\frac{1}{12})}\,\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}_{\eta}(0),$$

where the signs of the radicals are alike when $\eta' = \eta = 0$. Similarly, corresponding to the transformation T on η there is the transformation of integers

$$\left.\begin{array}{cc|cc} 1 & 2 & \multicolumn{2}{c}{\multirow{2}{*}{0}} \\ -1 & -1 & & \\ \hline 1 & 0 & -1 & 1 \\ 0 & -2 & -2 & 1 \end{array}\right.$$

* Kratzer-Prym, Neue Grundlagen einer Theorie der allgemeinen Thetafunctionen. pp. 121, 122.

on the theta moduli, and hence the relation

$$\frac{1}{\sqrt{1-\eta'^2}}\,\vartheta\begin{bmatrix} g_1+2g_2 & -g_1-g_2 \\ \frac{1}{2}-g_1-h_1+h_2 & 2g_2-2h_1+h_2 \end{bmatrix}_{\eta'}(0)$$

(2)

$$= \frac{\eta-\sqrt{2}}{\sqrt{1-\eta^2}}\,e^{-\pi i(g_1^2+4g_1g_2+2g_2^2-2g_2-g_1)}\,\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}_{\eta}(0),$$

where the signs of the radicals are taken *opposite* when $\eta'=\eta$. Again, to the transformation T_1 on η corresponds the transformation

$$\begin{array}{cc|cc} -1 & 2 & 2 & -1 \\ 1 & -1 & -1 & 1 \\ \hline -2 & 0 & 1 & -1 \\ 0 & -4 & -2 & 1 \end{array},$$

whence the relation

$$\frac{1}{\sqrt{1-\eta'^2}}\,\vartheta\begin{bmatrix} -g_1+2g_2-2h_1+h_2 & g_1-g_2+h_1-h_2 \\ 2g_1+h_1-h_2 & 4g_2-2h_1+h_2 \end{bmatrix}_{\eta'}(0)$$

(3)

$$= \frac{\sqrt{2}\,\varepsilon\,\eta-(\varepsilon+\varepsilon^3)}{\sqrt{1-\eta^2}}\,e^{-\pi iE}\,\vartheta\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}_{\eta}(0)$$

where

$$E = 2g_1^2-8g_1g_2+4g_2^2+8g_1h_1-4g_1h_2-8g_2h_1 + 8g_2h_2+4h_1^2-6h_1h_2+2h_2^2-\tfrac{1}{2}.$$

The signs of the radicals are alike when $\eta=\eta'$. Finally, corresponding to the identical substitution $\eta'=\eta$ there exists the unique table of integers

$$\begin{array}{cc|cc} 1 & 0 & & \\ 0 & 1 & 0 & \\ \hline & & 1 & 0 \\ & 0 & 0 & 1 \end{array}.$$

We introduce for brevity the notation

$$\tau\begin{bmatrix} g \\ h \end{bmatrix} = \frac{1}{\sqrt{1-\eta^2}}\,\vartheta\begin{bmatrix} g \\ h \end{bmatrix}_{\eta}(0)$$

and construct by the aid of (1), (2), (3) the tables $\left(j = e^{\frac{\pi i}{12}}\right)$

	$\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{6} & \frac{3}{2} \\ \frac{1}{6} & \frac{3}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{6} & -\frac{3}{2} \\ \frac{1}{6} & -\frac{3}{2} \end{bmatrix}$
S	$j^{-1}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$	$j^{2}\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$j^{-1}\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$	$j^{-4}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{6} & -\frac{3}{2} \\ \frac{1}{6} & -\frac{3}{2} \end{bmatrix}$	$\begin{bmatrix} \frac{1}{6} & \frac{3}{2} \\ \frac{1}{6} & \frac{3}{2} \end{bmatrix}$
T	$\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$	$j^{-3}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$j^{3}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$		
T_1	$j^{12}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$j^{-6}\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$	$j^{-6}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$	$j^{-4}\begin{bmatrix} \frac{1}{6} & -\frac{3}{2} \\ \frac{1}{6} & -\frac{3}{2} \end{bmatrix}$	$j^{-8}\begin{bmatrix} \frac{1}{6} & \frac{3}{2} \\ \frac{1}{6} & \frac{3}{2} \end{bmatrix}$

	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$
S	$j^{-1}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$	$j^{-1}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$	$j^{2}\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$	$j^{2}\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$	$j^{-1}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$j^{-1}\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$
T	$\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$	$j^{6}\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$	$j^{-3}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$	$j^{3}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$j^{-6}\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$
T_1	$j^{-6}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	$j^{6}\begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$	$j^{12}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}$	$j^{-6}\begin{bmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{bmatrix}$	$j^{-6}\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{bmatrix}$

In the first line of each is written $\tau\begin{bmatrix} g \\ h \end{bmatrix}$, and in the other lines the values when operated upon by the transformations S, T, T_1. For line three the factor $\eta' - \sqrt{2}$ is understood, for line four the factor $\sqrt{2}\,\varepsilon\,\eta' - (\varepsilon + \varepsilon^2)$.

It is necessary to show that the theta constants do not vanish identically. Let $\eta = (\varepsilon + \varepsilon^2 - i)/\sqrt{2}\,i$, the point A_1. The moduli become $\tau_{11} = i$, $\tau_{22} = 2\,i$, $\tau_{12} = \tau_{21} = (1 - i)$. Substituting these values in $\vartheta\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ we have after a simple reduction

$$\vartheta\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = i \sum_{n_1, n_2}^{-\infty, \infty} e^{-\pi\left[(n_1 + n_2 + 1)^2 + \left(n_2 + \frac{1}{2}\right)^2\right]}.$$

This result does not vanish since each term under the summation is positive; moreover, four thetas are completely interchanged by (1) and (2), and therefore none of them can vanish identically.

Let $\eta = 0$ and the moduli become $\tau_{11} = \varepsilon$, $\tau_{12} = \tau_{21} = 0$, $\tau_{22} = 2\varepsilon$. Substituting these values in $\vartheta\begin{bmatrix}0&0\\0&0\end{bmatrix}$ we have after reduction

$$\vartheta\begin{bmatrix}0&0\\0&0\end{bmatrix} = \sum_{n_1,n_2}^{-\infty,\infty} e^{-\frac{\pi i}{2}(n_1^2+2n_2^2)-\pi\frac{\sqrt{3}}{2}(n_1^2+2n_2^2)}.$$

Replacing n_1 by $2n_1$ and $2n_1+1$ and adding the results, the expression becomes

$$\sum e^{-\pi\sqrt{3}(2n_1^2+n_2^2)+\pi i n_2^2}\left(1+ie^{-\pi\sqrt{3}\left(2n_1+\frac{1}{2}\right)}\right).$$

Replacing now n_2 by $2n_2$ and $2n_2+1$ and combining results, we have finally

$$\sum e^{-\pi\sqrt{3}(2n_1^2+4n_2^2)}\left\{1+e^{-\pi\sqrt{3}(4n_2+1)}\right\}\left\{1+ie^{-\pi\sqrt{3}\left(2n_1+\frac{1}{2}\right)}\right\}.$$

It is clear that all the terms in the expansion are positive, whence the function does not vanish. Since six thetas are completely interchanged by (1), (2), it follows that none of them vanish identically.

By a similar process it may be shown that

$$\vartheta\begin{bmatrix}\frac{1}{6}&\frac{1}{3}\\\frac{1}{6}&\frac{2}{3}\end{bmatrix}, \qquad \vartheta\begin{bmatrix}\frac{1}{6}&-\frac{1}{3}\\\frac{1}{6}&-\frac{2}{3}\end{bmatrix}$$

do not vanish identically.

From the above tables, and what has just been proved, it is clear that

$$A(\eta) = \prod\begin{bmatrix}0&0\\0&\frac{1}{2}\end{bmatrix}\prod\begin{bmatrix}\frac{1}{2}&0\\0&\frac{1}{2}\end{bmatrix}\prod\begin{bmatrix}0&0\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}\prod\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix},$$

$$B(\eta) = \prod\begin{bmatrix}0&0\\0&0\end{bmatrix}\prod\begin{bmatrix}\frac{1}{2}&0\\0&0\end{bmatrix}\prod\begin{bmatrix}0&\frac{1}{2}\\0&0\end{bmatrix}\prod\begin{bmatrix}0&0\\\frac{1}{2}&0\end{bmatrix}\prod\begin{bmatrix}0&\frac{1}{2}\\\frac{1}{2}&0\end{bmatrix}\prod\begin{bmatrix}\frac{1}{2}&\frac{1}{2}\\0&0\end{bmatrix}$$

satisfy the relations

$$A(\varepsilon\eta) = \varepsilon A(\eta), \qquad A\left(\frac{\sqrt{2}i\eta - i}{i\eta - i\sqrt{2}}\right) = (i\eta - \sqrt{2}i)^4 A(\eta);$$

(4)

$$B(\varepsilon\eta) = B(\eta), \qquad B\left(\frac{\sqrt{2}i\eta - i}{i\eta - \sqrt{2}i}\right) = -(i\eta - \sqrt{2}i)^6 B(\eta),$$

and do not vanish identically. But (4) are exactly the functional equations (*a*), (*b*) of (6), § 3. Hence the functions $\psi_1(\eta)$, $\psi_2(\eta)$ are constant multiples of $A(\eta)$, $B(\eta)$, being relative invariants under G.

The function $\psi_3(\eta)$ of § 3 cannot be expressed as a product of theta functions with zero arguments. This function is of order 12, and has its separate theta constants invariant or interchanged in pairs under T since $T^2 = 1$. Under T the function $\tau\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix}$ becomes

$$\frac{i e^{\pi i E}}{(i\eta - \sqrt{2}\, i)}\, \tau\begin{bmatrix} g_1 + 2g_2 & -g_1 - g_2 \\ \frac{1}{2} - g_1 - h_1 + h_2 & 2g_2 - 2h_1 + h_2 \end{bmatrix}.$$

For invariance of the characteristic we have either

$$(a)\qquad \begin{aligned} g_1 + 2g_2 &\equiv g_1, & -g_1 - g_2 &\equiv g_2, \\ \tfrac{1}{2} - g_1 - h_1 + h_2 &\equiv h_1, & 2g_2 - 2h_1 + h_2 &\equiv h_2; \end{aligned}$$

or

$$(b)\qquad \begin{aligned} g_1 + 2g_2 &\equiv -g_1, & -g_1 - g_2 &\equiv -g_2, \\ \tfrac{1}{2} - g_1 - h_1 + h_2 &\equiv -h_1, & 2g_2 - 2h_1 + h_2 &\equiv -h_2. \end{aligned}$$

The only solutions of either of these sets of congruences which do not give identically vanishing theta constants are $\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. For each of these $E \equiv 0$. Thus $\tau\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ transforms into $\dfrac{i}{i\eta - \sqrt{2}\, i}\, \tau\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, and similarly for the other function $\tau\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$.

For a general characteristic

$$\tau\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix} \cdot \tau\begin{bmatrix} g_1 + 2g_2 & -g_1 - g_2 \\ \frac{1}{2} - g_1 - h_1 + h_2 & 2g_2 - 2h_1 + h_2 \end{bmatrix}$$

transforms into

$$\frac{-1}{(i\eta - \sqrt{2}\, i)^2}\, \tau\begin{bmatrix} g_1 & g_2 \\ h_1 & h_2 \end{bmatrix} \cdot \tau\begin{bmatrix} g_1 + 2g_2 & -g_1 - g_2 \\ \frac{1}{2} - g_1 - h_1 + h_2 & 2g_2 - 2h_1 + h_2 \end{bmatrix}.$$

A product of twelve such theta constants which is relatively invariant under G must be made up of powers of $\tau\begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $\tau\begin{bmatrix} 0 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$ and products of two thetas relatively invariant under T. In any case the product of *twelve* will be reproduced multiplied by $(i\eta - \sqrt{2}\, i)^{-12}$, whereas $\psi_3(\eta)$ has its multiplier under T *minus* this same expression.

Referring to equation (7) of § 3 we observe that $C \neq 0$. For if $C = 0$ or if $\psi_3(\eta) = 0$ then $\psi_1^6(\eta)/\psi_2^4(\eta)$ is constant. This is impossible since $\psi_1(\eta)$ vanishes at $\eta = 0$ and nowhere else in R, and $\psi_2(\eta)$ vanishes in R only for $\eta = \sqrt{2} - 1$. Hence, take $C = -1$, and we may determine A and B by placing in succession $\eta = 0$ and $\eta = \sqrt{2} - 1$, recalling at the same time that $\psi_3(\eta)$ vanishes at neither point. This gives an explicit form for $\psi_3^2(\eta)$.

From their definition it is now clear that

$$\frac{\psi_1^6(\eta)}{\psi_2^4(\eta)}, \quad \frac{\psi_1^6(\eta)}{\psi_3^2(\eta)}, \quad \frac{\psi_2^4(\eta)}{\psi_3^2(\eta)}, \quad \frac{a\,\psi_1^6(\eta) + b\,\psi_2^4(\eta)}{c\,\psi_1^6(\eta) + d\,\psi_2^4(\eta)},$$

a, b, c, d being constants, are *automorphic* functions of $(0, 3; 2, 4, 6)$. They are the simplest of such functions for the group. Moreover, every automorphic function for the given group can be expressed in the form

$$\prod_{i}^{n} \frac{A_i\,\psi_1^6(\eta) + B_i\,\psi_2^4(\eta)}{C_i\,\psi_1^6(\eta) + D_i\,\psi_2^4(\eta)}$$

where A_i, B_i, C_i, D_i are constants.

§ 5. **Uniformization.** In this section the problem of uniformizing the automorphic functions of the subgroups emphasized in 2 will now be studied.

The subgroup $(0, 3; 2, 6, 6)$.* From the tables of § 4 and the functional equation (8) of § 3 its is evident that

$$\Phi_2(\eta) = \eta \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & \frac{2}{3} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix} \eta \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{2}{3} \end{bmatrix}.$$

(1)

$$\Phi_2(\eta)\,\Phi_1(\eta) = \psi_1(\eta), \qquad \Phi_3(\eta) = \psi_2(\eta),$$

where $\psi_1(\eta)$, $\psi_2(\eta)$ are the functions defined in § 4. The function $\Phi_2(\eta)$ has been shown not to vanish identically, and hence $\Phi_1(\eta)$ does not. Reasoning as on p. 4 we deduce that a relation of the form

$$A_1\,\Phi_1^6(\eta) + B_1\,\Phi_2^6(\eta) + C_1\,\Phi_3^2(\eta) = 0 \tag{2}$$

must exist. Since only one of the three functions $\Phi_1(\eta)$, $\Phi_2(\eta)$, $\Phi_3(\eta)$ vanishes at the vertex of the fundamental region, it follows that none of the constants A_1, B_1, C_1 vanish.

* For additional facts regarding this subgroup see Hutchinson, l. c., § 4.

The results of applying the transformations S, T_1 are as follows:

$$(3)\quad \begin{aligned} S:\ \Phi_1(\eta') &= \varepsilon\Phi_1(\eta), \quad \Phi_2(\eta') = \Phi_2(\eta), \quad \Phi_3(\eta') = \Phi_3(\eta);\\ T_1:\ \Phi_1(\eta') &= -d^2\Phi_1(\eta), \quad \Phi_2(\eta') = -d^2\Phi_2(\eta), \quad \Phi_3(\eta') = d^6\Phi_3(\eta) \end{aligned}$$

where $d = \sqrt{2\varepsilon\eta - \varepsilon - \varepsilon^2}$. It is evident that

$$x = \frac{\Phi_1^6(\eta)}{\Phi_2^6(\eta)}, \qquad y = \frac{\Phi_3^2(\eta)}{\Phi_2^6(\eta)}$$

are the simplest *automorphic* functions belonging to the subgroup. We may now write equation (2) in the linear form

$$A_1 x + B_1 y + C_1 = 0,$$

which is a curve of genus *zero*, the coördinates x, y being expressed as uniform automorphic functions of η.

The subgroup (α). The simplest automorphic functions of this subgroup are the same as in the preceding case since T_1 enters *once* in each generating transformation. This leads to the same linear relation.

The subgroup g_1. We now build the quotients

$$(4)\quad y = \frac{\Phi_3(\eta)}{\Phi_2^3(\eta)}, \qquad x = \frac{\Phi_1(\eta)}{\Phi_2(\eta)}.$$

Operating with S and T_1 in turn on y and x we have

$$(5)\quad \begin{aligned} S:&\quad y' = y, \qquad x' = \varepsilon x,\\ T_1:&\quad y' = -y, \qquad x' = x. \end{aligned}$$

By referring to (3) § 3 it is seen that T_1 enters *twice* in each generating transformation; it follows from (5) that y is an automorphic function of the subgroup. Moreover, S enters a sufficient number of times in each generating transformation to make x absolutely invariant. To prove this statement for the transformation $T_1 S^{-1} T_1 S$ we notice that S introduces the factor ε, giving the result εx; T_1 leaves this unaltered, while $S^{-1} = S^5$ introduces the factor ε^5, making the result $\varepsilon^5 \cdot \varepsilon x = x$.

The function $\Phi_3(\eta)$ vanishes to the first order at the fixed point A_1 of T_1 and congruent points, and therefore it vanishes *six* times inside the fundamental region R'. Now $\Phi_2(\eta)$ vanishes to the first order at the fixed point A_7 of U_1 and the congruent points $A_8, A_9, \ldots, A_{12}$, all on the boundary of R'; but at each of these points there are *three* fundamental regions sharing this same zero, therefore at each such point $\Phi_2(\eta)$ has a zero of order $\frac{1}{3}$, and hence $\Phi_2(\eta)$ vanishes *twice* in R'. Finally, $\Phi_1(\eta)$ vanishes to the first order at the origin and the congruent points A_{13}, $A_{14}, \ldots, A_{18}$ on the boundary of R'; but at each of the latter points there are *six* fundamental regions sharing this same zero, whence at each such point $\Phi_1(\eta)$ has a zero of order $\frac{1}{6}$, so that altogether $\Phi_1(\eta)$ vanishes *twice* in R'.

It follows from the results under S in (5) that y takes the same value y_r at *six* different congruent points in R'; and that x takes the same value x_r at *one* point only, except when $x_r = 0$, when this value is taken at 0 and at six points on the boundary of the fundamental region.

Let the values of η at the fixed points of

$$T_1, \quad S T_1 S^{-1}, \quad S^2 T_1 S^{-2}, \quad \ldots. \quad S^5 T_1 S^{-5}$$

be

$$\eta_0, \quad \eta_1, \quad \ldots. \quad \eta_5,$$

and the corresponding ones for x.

$$x_0, \quad x_1, \quad x_2, \quad \ldots. \quad x_5.$$

Then we may put

$$y^2 = c(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5), \tag{6}$$

where c is a constant, for the two members are automorphic function of y_1 having the same *zero* and *poles* in R'. That the poles are the same for the two members is at once evident from the forms of y and x in (4). To show that the zeros are the same we observe that T_1 can be written in the form

$$\frac{\eta' - \eta_0}{\eta' - \eta_0'} = -\frac{\eta - \eta_0}{\eta - \eta_0'}$$

where η_0' is the other fixed point. Thus in applying T_1 to the point η_0 the expression $\frac{\eta - \eta_0}{\eta - \eta_0'}$ becomes $-\frac{\eta - \eta_0}{\eta - \eta_0'}$, and since x is unaltered by this operation $x - x_0$ must be 0^2 at $\eta = \eta_0$.

The values of $x_1, x_2, \ldots . x_5$ can be expressed in forms which enable us to write (6) in a more compact form. Apply to x the transformation S and we have $x(\varepsilon\eta) = \varepsilon x(\eta)$. Put $\eta = \eta_0$ and this becomes $x(\varepsilon\eta_0) = \varepsilon x(\eta_0)$, and in general $x(\varepsilon^i\eta) = \varepsilon^i x(\eta)$, whence $x(\varepsilon^i\eta_0) = \varepsilon^i x(\eta_0)$. That is $x_i = \varepsilon^i x_0$, $i = 1, 2, 3, 4, 5$. Hence (6) becomes

$$y^2 = c(x^6 - x_0^6). \tag{7}$$

We have thus the coördinates x, y of the *hyperelliptic* curve (7) expressed as automorphic functions of η. The constant c may be determined by putting $\eta = 0$. This gives $c = -\frac{y_0^2}{x_0^6}$ where y_0 is the value of y at $\eta = 0$.

The equivalent of relation (7) can at once be obtained from (2), but the above method is given in detail since it offers a means of uniformizing the functions when the zeros and poles are known in the fundamental region.

Since the subgroups $g_2, \ldots . g_5$ are equivalent to g_1 the same relation (7) holds for each of them.

The subgroup of genus 5. Several methods have been tried in an endeavor to uniformize the functions belonging to this group, but none so far have proved successful. It would seem that the coördinates of the curve of genus 5,

$$y^2 = c(x - x_1)(x - x_2)\cdots(x - x_{12})$$

are automorphic functions of this sub-group.

NOTE ON DIRICHLET'S SERIES WITH COMPLEX EXPONENTS.*

By Einar Hille.

1. **Introduction.** In the present paper we shall be chiefly concerned with *Dirichlet's series with complex exponents, i. e.* series of the form

$$D(s) = \sum a_n e^{-\lambda_n s} \tag{1}$$

where s is a complex variable and (λ_n) as well as (a_n) are sequences of complex numbers. We shall make various assumptions with regard to the absolute values of λ_n, but none whatever with regard to their arguments.† We shall be concerned chiefly with *absolutely convergent* series, and we generally assume $\sum |a_n|$ to be convergent, thus ensuring absolute convergence of (1) at one point at least, namely the origin.‡

* Presented to the American Mathematical Society, April 28th, 1923.

† The literature on Dirichlet's series with complex exponents appears to be rather meager. The following publications bear on the subject:

Schnee, Über irreguläre Potenzreihen und Dirichletsche Reihen, (Thesis, Berlin, 1908).

Pincherle, Sulla serie fattoriale generalizzate, Rendiconti del Circ. Mat. di Palermo, vol. 37, (1914), pp. 379–390.

Ritt, On a general class of linear homogeneous differential equations of infinite order with constant coefficients, (Thesis, Columbia, 1916), Transactions Amer. Math. Soc., vol. 18 (1917), pp. 27–49.

Väisälä, Verallgemeinerung des Begriffes der Dirichletschen Reihen, Acta Universitatis Dorpatensis, A I: 2, (1921), pp. 1–32.

Ritt makes the restriction that $\sum 1/\lambda_n$ is to be absolutely convergent but does not put any restrictions on the arguments of λ_n which the other writers on the field do. So for instance does Schnee assume the existence of $\lim \arg \lambda_n$ whereas Pincherle assumes $\vartheta < \arg \lambda_n < \vartheta + \pi$ for some fixed value of ϑ. On the other hand, these authors study simple convergence whereas in the present paper we have to restrict ourselves in general to the study of absolute convergence.

Väisälä assumes $\lim \arg \lambda_n = 0$ when studying simple convergence but has also some results concerning absolutely convergent series where no restriction is put on $\arg \lambda_n$. The present writer was in possession of these and also more general results before the investigation of Väisälä appeared in print. The results in question have been incorporated with the present paper, forming §§ 3 and 6 and part of § 5. The proofs here given are somewhat simpler than those of Väisälä.

‡ It is worth noticing that an ordinary Dirichlet's series (λ_n real, $\lambda_n < \lambda_{n+1}$, $\lambda_n \to +\infty$) is absolutely convergent at one point at least, namely $s = +\infty$.

We prove that *the region of absolute convergence of* (1) *is a convex region.* We attempt to determine this region in terms of (a_n) and (λ_n) under various assumptions regarding the rate of increase of $|\lambda_n|$. If $|\lambda_n|/\log n \to \infty$ with n, then we can determine the precise region. In any case we can find a *maximal region* and a *minimal one.* It is necessary for a point to remain in the former in order to make the series convergent and it is sufficient for absolute convergence that it stays in the latter. To any given convex region we can determine infinitely many Dirichlet's series which will admit of that region as their region of absolute convergence. We also give a *comparison theorem* for Dirichlet's series which allows certain interesting conclusions regarding the singular points of the analytic function represented by the series.

Further we extend some of the results to *Dirichlet's integrals.* Finally we take up the connections between generalisations of *Dirichlet's series, binomial series* and *factorial series.*

The methods used in the paper are essentially elementary in character and the results can certainly be extended considerably.

2. **Auxiliary theorems.** In this paragraph we collect a few lemmas and definitions which we shall use in the paper proper. The proofs are in general omitted but references are made to the literature when necessary.

DEFINITION: *A region is said to be convex if, whenever it contains two points P_1 and P_2, it also contains all the points of the linear segment $P_1 P_2$.*

LEMMA 1: *If a and b are real positive numbers and $0 \leq x \leq 1$, then*

$$a^x \cdot b^{1-x} \leq x \cdot a + (1-x) \cdot b \leq a + b,$$

where the equality sign holds always when $a = b$, otherwise only for $x = 0$ or 1.

The proof follows from the fact that $\frac{d^2}{dx^2}[a^x b^{1-x}] > 0$. From this lemma we obtain

LEMMA 2: *If the series $\sum a_n$ and $\sum b_n$ are convergent, where $a_n, b_n \geq 0$, then the series*

$$\sum a_n^x b_n^{1-x} \qquad (0 \leq x \leq 1)$$

are also convergent, and uniformly with respect to x.

LEMMA 3: *If (b_n) and (c_n) are two sets of complex numbers and if the series $\sum b_n$ and $\sum |c_{n+1} - c_n|$ are convergent, then the series $\sum b_n c_n$ converges.**

* For a proof of this lemma see *e. g.* Landau, Über die Grundlagen der Theorie der Fakultätenreihen, Münchener Sitzungsberichte, Math.-Phys. Klasse, vol. 36 (1906), p. 155.

LEMMA 4: *If the series* $\sum u_n$ *with positive terms is convergent and if* $R_n = \sum_n^\infty u_m$, *then the series*

$$\sum \frac{u_n}{R_n^{1-\alpha}}$$

is convergent provided $\alpha > 0$.*

A similar theorem holds for definite integrals, namely

LEMMA 5: *Let* $a(t)$ *be a real positive integrable function such that the integral* $\int_0^\infty a(t)\,dt$ *exists and put* $R(t) = \int_t^\infty a(u)\,du$, *then the integral*

$$\int_0^\infty \frac{a(t)\,dt}{[R(t)]^{1-\alpha}}$$

exists, provided the real number α *is positive.*

This lemma can of course be proved by merely making the substitution $r = R(t)$ in the integral in question.

3. **The region of absolute convergence.** Using Lemma 2 we shall prove the following

THEOREM 1: *The region of absolute convergence of a Dirichlet's series is a convex region* R. *The series is uniformly convergent in every interior part of* R. *If* R *is a two-dimensional region the series represents a single-valued and analytic function of* s *in* R.

Suppose that the series is absolutely convergent for two values of s, namely s_1 and s_2, and let us put

$$|a_n e^{-\lambda_n s_1}| = A_n, \quad |a_n e^{-\lambda_n s_2}| = B_n.$$

Any point on the line segment $s_1 s_2$ can be represented in the form

$$s = \alpha s_1 + (1-\alpha) s_2. \qquad (0 \leq \alpha \leq 1).$$

Then we have

$$|a_n e^{-\lambda_n s}| = A_n^\alpha B_n^{1-\alpha}$$

which shows that the series is absolutely convergent at every point of the line segment $s_1 s_2$, and, moreover, that the convergence is uniform with respect to s. The rest of the theorem follows by classical arguments.†

* Cf. P. du Bois-Reymond, Eine neue Theorie der Convergenz und Divergenz von Reihen mit positiven Gliedern, Journ. f. Math., vol. 76 (1873), p. 85.

† When this paper was presented to the Society, Professor Ritt pointed out to me that he had used the same argument to prove practically the same question in his thesis, quoted above. I was previously unaware of the existence of his paper. The theorem and the proof had been in my possession since 1918.

It is worth noticing that the statement: The region of convergence of a Dirichlet's series is a convex region, would be *false*. A simple example will show that. According to our definition, an ordinary trigonometric series is a Dirichlet's series. The region of convergence of such a series may reduce to a set of points of measure zero on the real axis.* As a matter of fact, Schnee pointed out† that it was necessary to restrict the problem and gave an example of a Dirichlet's series which had an isolated point of convergence in addition to the ordinary half-plane. The restrictions he laid on the problem were to limit the variability of arg λ_n whereas we have taken the more elementary point of view of studying absolute convergence only.

4. **The maximal region.** In this paragraph we shall assume $\sum a_n$ to be convergent, *i. e.* we assume the series (1) to be convergent at the origin. With this assumption we shall determine a necessary condition to be fulfilled by s in order that the series may be convergent for that value. In the case of an ordinary Dirichlet's series with real exponents, $\lambda_n < \lambda_{n+1}$, $\lambda_n \to +\infty$ we get such a condition in the following manner.‡ Let us put

$$\pi_n = \frac{\log |a_n|}{\lambda_n} \tag{2}$$

and let π' be the upper limit of the π_n,

$$\pi' = \overline{\lim}\, \pi_n. \tag{3}$$

Then in order that the series be convergent it is necessary that $\Re(s) \geq \pi'$, or if we assume $\pi' < 0$,

$$\Re\left(\frac{s}{\pi'}\right) \leq 1. \tag{4}$$

If $\lambda_n/\log n \to +\infty$, the condition $\Re\left(\frac{s}{\pi'}\right) < 1$ is sufficient for absolute convergence§, if $\lambda_n = \log n$ the abscissa of absolute convergence, $\bar{\sigma}$ satisfies the inequalitiy $\pi' \leq \bar{\sigma} \leq \pi' + 1$.

In the case when the λ_n are complex numbers we proceed as follows. Let the set (π_n) be defined by (2) and let (π') be the set of its limit

* See, *e. g.* Kolmagoroff, Une série de Fourier-Lebesgue divergent presque partout, Fundamenta Mathematicae, vol. 4 (1923), pp. 324–328.

† Loc. cit., pp. 18–19.

‡ Cf. Landau, loc. cit. pp. 177–178 for fuller references.

§ Cf. Szász, Über Singularitäten von Potenzreihen und Dirichletschen Reihen am Rande des Konvergenzbereiches, Math. Annalen, vol. 85 (1922), p. 108.

points.* Take any point, $\pi' \neq 0$, of the latter set and draw a straight line through the point perpendicular to the line joining 0 with π'. This line divides the plane into two parts, let the part containing the origin be denoted by $\Delta(\pi')$. It is defined by the inequality

$$\Re\left(\frac{s}{\pi'}\right) \leq 1. \tag{5}$$

We repeat this construction for all the points of (π'). If the origin happens to be a point of (π') we obtain the corresponding region or regions $\Delta(0)$ by a limiting process. There exists in that case a set of points in (π_n), $\pi_{11}, \pi_{21}, \cdots, \pi_{m1}, \cdots$ such that $\lim_{m\to\infty} \pi_{m1} = 0$. Form the corresponding half-planes $\Delta(\pi_{m1})$. This set of half-planes may or may not have a definite limit. In the former case we adjoin this limit $\Delta(0)$ to the set of half-planes $\Delta(\pi')$ defined above. In the latter case we can pick out a sub-set that will have a definite limit which we adjoin instead. Then we pick out all possible sub-sets from the given set, adjoining the corresponding limiting half-planes. Likewise we exhaust all possible sets (π_{mk}) such that $\lim_{m\to\infty} \pi_{mk} = 0$, adjoining all the corresponding half-planes $\Delta(0)$. In this fashion we obtain a set of half-planes $\Delta(\pi')$ namely one and only one, containing the origin as an *interior* point, for each of the points of $\{\pi'\}$ different from zero, and, eventually, in addition a sub-set, which may have the power of the continuum, having zero as a *boundary* point. Let Δ be the set of all points common to all half-planes of the set $\{\Delta(\pi')\}$. This region Δ will contain one point at least, namely the origin, and may reduce to that point; at any rate, from its construction, it will be a *closed convex region*. We designate Δ as *the maximal region of convergence*; the reason for the name appears from the following

THEOREM 2: *The series* (1) *is divergent for every value of* s *not in* Δ.

As matter of fact, if s is not a point in Δ the inequality

$$\Re\left(\frac{s}{\pi'}\right) > 1$$

is fulfilled for at least some point of (π'). In other words, there exists a sub-set of points in (π_n), $\pi_{n1}, \pi_{n2}, \cdots, \pi_{nm}, \cdots$ such that

$$\Re\left(\frac{s}{\pi_{nm}}\right) = \Re\left(\frac{s\lambda_{nm}}{\log|a_{nm}|}\right) > 1.$$

*We include in (π') any point π_0 such that $\pi_n = \pi_0$ for infinitely many values of n.

Since the series $\sum a_n$ is convergent we know that $a_n \to 0$. Hence the inequality

$$\Re(s\lambda_n) < \log |a_n|$$

is fulfilled for values of n beyond every limit. This implies

$$\Re[\log |a_n| - s\lambda_n] > 0,$$

or

$$|a_n e^{-\lambda_n s}| > 1$$

for arbitrarily large values of n. Hence the series is divergent outside of Δ.

5. **Special cases.** We notice that the region of convergence R of (1) as well as the region of absolute convergence $\bar{R}$ form parts of Δ, or symbolically

$$\bar{R} \subseteq R \subseteq \Delta. \tag{6}$$

There is one case, however, in which we can give a much more precise statement, namely, that in which the absolute values of the λ_n increase more rapidly than $\log n$. Denoting the boundary of the closed region Δ by Γ we have the following theorem.

THEOREM 3: *If* $|\lambda_n|/\log n \to \infty$ *as* n *becomes infinite then the series* (1) *is absolutely convergent in the open region* $\Delta - \Gamma$. *The series may or may not be convergent or absolutely convergent on* Γ *or on parts of* Γ.

The theorem is of interest only if Δ is a two-dimensional region since otherwise the region $\Delta - \Gamma$ is empty. Then we can assume that 0 is a point of $\Delta - \Gamma$ without restricting the generality of the argument. Hence

$$|\pi_n| \geqq \epsilon > 0, \; n \geqq n_0.$$

Further

$$|\lambda_n| = \frac{\log \dfrac{1}{|a_n|}}{|\pi_n|}$$

when $|a_n| < 1$, or

$$\log \frac{1}{|a_n|} = |\lambda_n| \, |\pi_n| > \epsilon |\lambda_n|.$$

Since $|\lambda_n|/\log n \to \infty$ we can make

$$|\lambda_n| > \frac{2}{\epsilon\delta} \log n, \; n \geqq n_0.^*$$

* The factor 2 is of course arbitrary; any fixed positive number greater than 1 would do just as well.

no matter how small a positive number δ may be, provided we make n_0 large enough. Hence

$$\log \frac{1}{|a_n|} > \frac{2}{\delta} \log n,$$

or

$$|a_n| < \left(\frac{1}{n}\right)^{\frac{2}{\delta}}.$$

If s is a point in $\Delta - \Gamma$, we can find a positive δ such that

$$1 - \Re\left(\frac{s}{\pi_n}\right) \geqq \delta,$$

when $n \geqq n_0$. Thus

$$|a_n e^{-\lambda_n s}| = |a_n|^{1-\Re\left(\frac{s}{\pi_n}\right)} < \left(\frac{1}{n}\right)^{\frac{2}{\delta}\cdot\delta} = \frac{1}{n^2},$$

which shows that the series is absolutely convergent in $\Delta - \Gamma$.

Let us make the less restrictive assumption that

$$|\lambda_n| > C \log n.$$

Then as above we find

$$|a_n| < \left(\frac{1}{n}\right)^{C\varepsilon},$$

and

$$|a_n e^{-\lambda_n s}| < \left(\frac{1}{n}\right)^{C\varepsilon\left[1-\Re\left(\frac{s}{\pi_n}\right)\right]}.$$

Hence, if we require s to satisfy all the conditions

$$\Re\left(\frac{s}{\pi'}\right) < 1 - \frac{1+\eta}{C\varepsilon}, \quad \eta > 0,$$

the series will be absolutely convergent for such a value of s. This result is of interest only when $C\varepsilon > 1$. If this condition is fulfilled we can determine a set of half-planes $\Delta(\pi')$ by the inequality

$$\Re\left(\frac{s}{\pi'}\right) < 1 - \frac{1}{C\varepsilon}. \tag{7}$$

Let $\bar{\Delta}$ be the set of all points common to all the half-planes $\bar{\Delta}(\pi')$. This is an open convex region interior to Δ.

THEOREM 4: *If* $|\lambda_n| \geq C \log n$ *and the minimum distance of* Γ *to the origin,* ϵ, *is so large that* $C\epsilon > 1$, *then the series* (1) *is necessarily absolutely convergent in the open convex region* Δ *defined above.*

This is the best result we can expect to obtain in view of the situation in which $\lambda_n = \log n$. mentioned above in § 4.

6. **Preassigned region of absolute convergence.** The boundary Γ of Δ is a continuous curve. Sometimes the set (π') also makes up a continuous curve H which is the *pedal* of Γ. By extension we shall always refer to the set (π') as the pedal of the boundary. This observation together with Theorem 3 enables us to solve the following problem:

Given an arbitrary open convex region, find a Dirichlet's series which will admit of this region as its region of absolute convergence.

We assume the region in question, Δ, to be two-dimensional (the one-dimensional case being trivial) and that 0 is a point of it. Further we shall restrict the boundary curve Γ to be of *finite length*, L. The case when that is not so, can be dealt with by a method similar to the one which we are going to develop. For the sake of brevity we shall exclude this case, however. Let l denote the length of arc of the curve measured from a fixed point in positive sense to a variable point, and put

$$l/L = \alpha. \tag{8}$$

Suppose that the equation of Γ is given in the form

$$s = f(\alpha), \; f(0) = f(1). \tag{9}$$

This continuous function of α can be defined for all values of α as a *periodic* function with period 1. The curve Γ has a unique tangent *almost everywhere*, *i. e.* excepting at most a point set of measure zero, and the function $f(\alpha)$ has a derivative almost everywhere. The equation of the pedal H of this curve Γ becomes

$$s = F(\alpha) = -\frac{i}{L^2} f'(\alpha) \, \Im \left[f'(\alpha) \overline{f(\alpha)} \right] \tag{10}$$

where $\overline{f(\alpha)}$ is the conjugate of $f(\alpha)$.

This function $F(\alpha)$ is continuous almost everywhere and the discontinuities are of the first kind only, *i. e.* the limits $\lim F(\alpha + \epsilon)$ and $\lim F(\alpha - \epsilon)$ exist and are finite. Now let θ be an *irrational number* and put

$$\pi_n = F(n\theta). \tag{11}$$

If this expression should be indeterminate for some particular value of n, we take $\lim F(n\theta - \varepsilon)$ which does exist.

Given any particular point π' on the pedal Π we can find a sequence of values of n, n_ν, such that $F(n_\nu \theta) \to \pi'$ when $\nu \to \infty$. This depends upon the fact that given any number g, $0 \leq g \leq 1$, we can find integers $n_1, n_2, \cdots n_\nu, \cdots$ such that

$$(g - n_\nu \theta) \to 0$$

when $\nu \to \infty$. Here (x) stands for the difference between x and the nearest integer. We notice that

$$\min |f(\alpha)| \leq |\pi_n| \leq \max |f(\alpha)|.$$

Now let us take

$$a_n = e^{-n}, \qquad \lambda_n = -\frac{n}{F(n\theta)}. \tag{12}$$

The resulting series will have the given region Δ as region of absolute convergence as evidently $|\lambda_n|/\log n \to \infty$. This is of course not the only such series. As a matter of fact, we have a great latitude of choice in a_n and also in λ_n.

7. **The minimal region.** In § 4 we found necessary conditions for the convergence of the series (1) assuming the convergence of $\sum a_n$. The conditions were found to be sufficient to ensure even absolute convergence provided we added a rather restrictive assumption on the rate of increase of $|\lambda_n|$. Since we are not able to give necessary and sufficient conditions which will work in every case, it remains for us to give at least *sufficient conditions for absolute convergence* of (1) which do not involve restrictions on λ_n. We can give such conditions if we assume the convergence of $\sum |a_n|$, *i. e.* in the case in which the series (1) is known *a priori* to be absolutely convergent at one point, namely the origin.

Putting

$$R_n = \sum_n^\infty |a_m|, \tag{13}$$

we know that $R_n \to 0$ when $n \to \infty$. Let us form

$$p_n = \frac{\log R_n}{\lambda_n}. \tag{14}$$

This set has a set of limiting points (p') on which we build the set of half-planes $\{D(p')\}$ in a manner similar to that in which the set of half-

planes $\{\Delta(\pi')\}$ was formed on the point set (π') above in § 4; the only difference being that the half-plane $D(p')$ is defined by

$$\Re\left(\frac{s}{p'}\right) < 1, \tag{15}$$

whereas in equation (5) we allowed ≤ 1. The set of points common to all the half-planes $D(p')$ forms an *open convex region* D. We designate D as *the minimal region of absolute convergence* of the series (1). It is evident that D is a part of Δ:

$$D \subset \Delta. \tag{16}$$

As a matter of fact, $R_n > |a_n|$ from which it follows that

$$\arg p_n = \arg \pi_n; \qquad |p_n| < |\pi_n|.$$

Further, we can group the half-planes in the set $\{D(p')\}$ in subsets D_θ, where the half-planes in D_θ are characterized by

$$\arg p' = \theta, \qquad (-\pi < \theta \leq +\pi).$$

We can arrange the half-planes $\{\Delta(\pi')\}$ in the same manner. A simple consideration shows that we can establish a one-to-one correspondence between these two sets of sub-sets by letting D_θ correspond to Δ_θ; moreover, the half-plane in Δ_θ which has the least distance from the origin will not be nearer to the origin than the half-plane in D_θ which has minimum distance. This shows that the region D is obtained from Δ by a certain contraction with the origin as a fixed point.

THEOREM 5: *If* $\sum |a_n|$ *is convergent the series* (1) *is absolutely convergent in the open convex region* D *defined above.*

If s is a point in D then the inequality

$$\Re\left(\frac{s}{p_n}\right) \leq 1 - \delta, \qquad \delta > 0,$$

is fulfilled from a certain value of n on. In view of (14) this inequality can be written

$$\Re\left(\frac{s\,\lambda_n}{\log R_n}\right) \leq 1 - \delta,$$

or

$$-\Re(s\,\lambda_n) \leq -(1-\delta)\log R_n.$$

Hence

$$|e^{-\lambda_n s}| \leqq \frac{1}{R_n^{1-\delta}},$$

or

$$|a_n e^{-\lambda_n s}| \leqq \frac{|a_n|}{R_n^{1-\delta}}.$$

Thus by Lemma 4 the series (1) is absolutely convergent in D.

Sometimes this region D will coincide with $\Delta - \Gamma$. This will be the case if for instance

$$R_n = |a_n|[1 + o(1)]. \tag{17}$$

A much less stringent condition is furnished by

$$\log R_n = \log|a_n| + o(|\lambda_n|). \tag{18}$$

This latter condition is fulfilled *e. g.* when a_n is a steadily decreasing logarithmico-exponential function of n and $|\lambda_n|/\log n \to \infty$. One might feel inclined to believe that in the case in which $|\lambda_n|/\log n \to \infty$, the region D would be the precise region of absolute convergence, at any rate if the λ_n are ordered according to increasing absolute values. This is however not the case as is shown by the following example.

We define a_n and λ_n in the following manner:

If

$$n \neq k^2, \qquad a_n = 2^{-n}, \qquad \lambda_n = +n \log 2;$$

if

$$n = k^2, \qquad a_n = 2^{-n^2}, \qquad \lambda_n = -n \log 2.$$

Thus we are dealing with the sum

$$\sum{}' 2^{-n-ns} + \sum 2^{-k^2+k^2 s}, \tag{19}$$

where the prime on the first summation sign indicates that n is different from a square number. By inspection we find that the first series is convergent if $\Re(s) > -1$, and the second series is always convergent. Thus the sum is convergent if $\Re(s) > -1$. The same result we get by applying Theorem 3. Let us now determine the region D. A simple calculation shows that $R_n = 2^{-n} k_n$, where $\frac{1}{2} < k_n < 2$, for every value of n. If $n \to \infty$, running through the square numbers, $p_n \to +1$; the only other limiting value of p_n is of course -1. Hence D consists of merely the

infinite strip $-1 < \Re(s) < +1$. It is evidently the irregular decrease of the a_n-sequence which accounts for this poor result.

8. **A comparison theorem.** In his thesis* Harald Cramér gave a theorem which involves a comparison of the region of holomorphism of two Dirichlet's series related in a particular fashion. We shall extend this theorem to hold for absolutely convergent Dirichlet's series with complex exponents. As a matter of fact, it is not necessary to assume absolute convergence of the series, uniform convergence in any finite closed part of the region under consideration is all we require for the proof.

THEOREM 6: *Let* $D(s) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}$ *be a Dirichlet's series which is absolutely convergent in a certain open region* R *containing the circular domain* $|s| \leqq d$.

Take the entire function $E(z) = \sum_{\nu=0}^{\infty} \frac{c_\nu}{\nu!} z^\nu$, *satisfying the condition* $|E(z)| < e^{(k+\varepsilon)|z|}$ *when* $|z| > \varrho$ *for any positive* ε, *where* $k < d$, *and form the composed series*

$$D_E(s) = \sum_{n=0}^{\infty} a_n E(\lambda_n)\, e^{-\lambda_n s}. \tag{20}$$

Let R_k *be the set of points in* R *whose distance from the boundary of* R *is greater than* k. *Then the series* $D_E(s)$ *is absolutely convergent in* R_k *at least.*

Suppose that the function represented by the series $D(s)$ *is holomorphic in a certain region* D, *containing* R. *This region may be in the plane or on a Riemann surface. Let* D_k *be the region formed by all points in* D *whose distance from the boundary of* D *is greater than* k. *If* D_k *should consist of several disconnected pieces, let* D_k^* *be the piece which contains* R_k.

Then the function represented by $D_E(s)$ *is holomorphic in* D_k^*. *If, in particular,* $k = 0$, *as is the case when* $E(z)$ *is of genus zero or of genus one and of minimal type* (in Pringsheim's notation) *the function* $D_E(s)$ *will have no other singular points than those of* $D(s)$.

The condition that the circle $|s| = d$ shall be contained in R ensures the existence of points in R_k and can, of course, be modified. The proof

* Sur une classe de séries de Dirichlet, Uppsala (1917) pp. 30–34, where the theorem is proved for absolutely convergent series. Extension to simply convergent series and various interesting applications are given in Un théorème sur les séries de Dirichlet et son application, Arkiv för Matematik etc., vol. 13, No. 22 (1918). These papers will be referred to by C 1 and C 2 respectively in this paragraph. A simpler proof has been given by Ostrowski in Einige Bemerkungen über Singularitäten Taylorscher und Dirichletscher Reihen, Berl. Sitz.-Ber., Phys.-Math. Klasse, 1923, pp. 39–44.

given by Ostrowski for the case in which the exponents are real numbers applies without any changes to the case of complex exponents. For the sake of completeness we shall reproduce the proof, however.

From the assumption on the rate of increase of $E(z)$ it follows that

$$\overline{\lim}\ \sqrt[n]{|c_n|} \leqq k.$$

Hence the power series

$$E^*(z) = \sum_{n=0}^{\infty} (-1)^n \frac{c_n}{z^{n+1}}$$

converges when $|z| > k$. Let us form the integral

$$\frac{1}{2\pi i}\int D(s+u)\ E^*(u)\ du, \tag{21}$$

where s lies in D_k^* and we integrate along the circle $|u| = k+\varepsilon$ with such a small positive value for ε that $s+u$ will always stay in D. This integral represents a function which is holomorphic in D_k^*. In order to represent it in D_k^* we use the series for $E^*(z)$ and integrate term-wise. By Cauchy's theorem the value of the integral is

$$\sum_{\nu=0}^{\infty} (-1)^\nu \frac{c_\nu}{\nu!} D^{(\nu)}(s).$$

If on the other hand we use the Dirichlet's series for $D(s+u)$ for a value of s in R_k and integrate term-wise we have

$$\frac{1}{2\pi i}\int D(s+u)\ E^*(u)\ du = \sum_{n=0}^{\infty} a_n\ E(\lambda_n)\ e^{-\lambda_n s} = D_E(s), \tag{22}$$

which completes the proof of the theorem.

In C2 Cramér gives various applications of his comparison theorem, some of which can be carried over to Dirichlet's series with complex exponents. We shall take up a few of these applications. Cramér's leading idea can be expressed as follows (cf. C2, pp. 8–9). Suppose that $D(s)$ can be expanded in a Laurent series, convergent for $0 \leqq r_1 < |s-s_0| < r_2$, which contains at least one negative power of $(s-s_0)$, and suppose that R has a part A in common with this circular ring. Then if the number k is so chosen that (i) $k < \frac{1}{2}(r_2-r_1)$, and (ii) A has a part A_k in common with R_k, then the circular ring $r_1+k < |s-s_0| < r_2-k$ forms a part of the region D_k^* and, moreover, *the function $D_E(s)$ has at least one singular point on or within*

the circle $|s-s_0| = r_1 + k$. This depends upon the fact that $D_E(s)$ can also be expanded in a convergent Laurent series which contains at least one negative power of $(s-s_0)$.

This observation can evidently be extended in various directions. The following extension will be of importance to us later. Instead of taking a circular ring we may take the region between two simple closed non-intersecting Jordan curves C_1 (interior) and C_2 (exterior). Let s_0 be any point inside of C_1. Then we can find an analytic function $w = f(s) = \sum_{n=1}^{\infty} b_n (s-s_0)^n$, $b_1 > 0$, which maps the interior of C_2 conformally on the interior of the unit-circle in the w-plane. Any function $D(s)$, single-valued and analytic in the annular ring between C_1 and C_2 can be expanded in a convergent Laurent series in $w = f(s)$ which will contain negative powers if $D(s)$ admits of a singular point on or within C_1. We identify this function $D(s)$ with the function represented by the Dirichlet's series (1) and compose this series with an entire function $E(z)$. If the number k is sufficiently small the resulting series $D_E(s)$ will be represented by a Laurent series in w convergent in some annular ring formed by two curves Γ_1 and Γ_2 between C_1 and C_2, and $D_E(s)$ will have at least one singular point on or within Γ_1.

As a first application of this principle we find (cf. C2, pp. 10–11): Given a set of complex numbers $\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots, |\lambda_n| \leq |\lambda_{n+1}|$, and suppose that we can form a Dirichlet's series $D(s) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n s}$ such that the function $D(s)$ has a singular point s_0 on the boundary of the region of convergence, in the neighborhood of which $D(s)$ can be expanded in a Laurent series convergent for $0 < |s-s_0| < r$. Then *every entire function* $E(z)$ *such that* $|E(z)| < e^{k|z|}$ *where* $k < \frac{1}{2} r$, *which vanishes for* $z = \lambda_n$ $(n = 1, 2, \ldots)$ *will vanish identically.*

This result has a direct application to the theory of Dirichlet's series such that $\sum 1/|\lambda_n|$ is convergent. Such series were the object of Ritt's paper quoted above.* He showed that *such a series represents a function single-valued and analytic, together with all its integrals, throughout its region of existence,* and, moreover, that *the analyticity of the function along the circumference of any circle in the plane requires its analyticity inside the circle.* We can go one step further, namely the circle can be replaced by any simple closed Jordan curve; in other words, *the region of existence, W, of the function represented by the series is a simply-connected region.*

This follows easily from the extension of Cramér's fundamental idea mentioned above. If we suppose the region W to be multiply-connected,

* Loc. cit. p. 38, theorems XI and XII. Also consult Cramér, C 2, p. 11, who restricts himself to real values of λ_n.

there exist simple closed Jordan curves in W which cannot be contracted to a point without leaving the region. Take two such curves C_1 and C_2, one interior to the other. Then if $D(s)$ is composed with an entire function $E(z)$ of genus zero the resulting series $D_E(s)$ will necessarily have a singular point inside of C_1. If, however, we take $E(z) = \prod_{n=1}^{\infty}\left(1-\frac{z}{\lambda_n}\right)$, the resulting series $D_E(s)$ *is identically zero* which involves a contradiction. The boundary of W may of course reduce to the point at infinity. Examples of this possibility are given by Ritt and Cramér in their papers.

9. **Dirichlet's integrals.** The theorems we have proved for generalized Dirichlet's series can be extended to generalized Dirichlet's integrals. We shall only give the necessary definitions and indicate some of the extensions.

We define two single-valued complex functions $a(t)$ and $\lambda(t)$ of the real variable t on the range $(0, +\infty)$. We assume these functions to be bounded in absolute value on every finite interval $(0, \omega)$. The integral $\int_0^{\omega} |a(t)|\,dt$ shall exist for every finite value of ω. Further we assume that $|\lambda(t)| \to \infty$ when $t \to \infty$. Consider the integrals

$$I(s) = \int_0^{\infty} a(t)\, e^{-\lambda(t)s}\, dt, \tag{23}$$

$$J(s, \omega) = \int_0^{\omega} |a(t)\, e^{-\lambda(t)s}|\, dt. \tag{24}$$

If $\lim_{\omega \to +\infty} J(s, \omega)$ is finite for a definite value s_1 of s, the integral $I(s)$ is said to be *absolutely convergent* for $s = s_1$, otherwise absolutely divergent. Then we have

THEOREM 7: *The points of the s-plane for which the integral $I(s)$ is absolutely convergent form a convex domain.*

This theorem is an immediate consequence of Lemma 1. In fact, if $0 \leq \alpha \leq 1$ we have

$$J\{\alpha s_1 + (1-\alpha)s_2, \omega\} \leq \alpha J(s_1, \omega) + (1-\alpha) J(s_2, \omega).$$

Hence if $\lim_{\omega \to +\infty} J(s, \omega)$ exists for $s = s_1$ and s_2, it will also exist for $s = \alpha s_1 + (1-\alpha)s_2$, which is all we have to show.

In theorem 2 we merely used the fact that the general term of a convergent series approaches zero. There is no analogue in the theory of convergent integrals; quite the contrary, the integrand may show arbitra-

rily high peaks without jeopardizing the convergence. We can obtain an analogue of theorem 2, but only by making rather restrictive assumptions which considerably reduce the interest of the result. The only theorem which extends easily to Dirichlet's integrals is Theorem 5 concerning the minimal region of absolute convergence.

Let us suppose the integral $\int_0^\infty |a(t)|\,dt$ to be convergent and put

$$R(\omega) = \int_\omega^\infty |a(t)|\,dt, \tag{25}$$

forming

$$p(\omega) = \frac{\log R(\omega)}{\lambda(\omega)}. \tag{26}$$

Let $P(\Omega)$ be the set of values taken on by $p(\omega)$ when $\omega \geqq \Omega$ and let further (p) be the set of values common to all the sets $P(\Omega)$. Finally, if the set (p) is not closed, add to it the missing limiting values; let (p') be the closed set so formed. On the set (p') we form an open convex region D by exactly the same process as in § 7. Then we have:

THEOREM 8: *If the integral* $\int_0^\infty |a(t)|\,dt$ *is convergent, the integral* $I(s)$ *is absolutely convergent in the open convex region* D *defined above.*

The proof is the same as that of theorem 5, except that Lemma 5 takes the place of Lemma 4.

10. **Factorial series and binomial series.** It is well known that there exists an intimate relation between *Dirichlet's series*, *factorial series* and *binomial series*. Let us put

$$F(s) = \sum a_n \frac{\alpha_1\,\alpha_2 \cdots \alpha_n}{(s+\alpha_1)(s+\alpha_2)\cdots(s+\alpha_n)}, \tag{27}$$

$$B(s) = \sum (-1)^n a_n \frac{(s-\alpha_1)(s-\alpha_2)\cdots(s-\alpha_n)}{\alpha_1\,\alpha_2 \cdots \alpha_n}, \tag{28}$$

where the α_n are complex numbers such that the series $\sum \frac{1}{\alpha_n}$ is divergent whereas the series $\sum \frac{1}{|\alpha_n|^2}$ converges. We put

$$\sum_{m=1}^{n} \frac{1}{\alpha_m} = \lambda_n \text{ and } D(s) = \sum a_n e^{-\lambda_n s}. \tag{29}$$

THEOREM 9: *The series* $B(s)$, $D(s)$ *and* $F(s)$ *are convergent (absolutely convergent) for the same values of* s. *The only points for which this state-*

ment may fail are the values $\pm \alpha_n$, *the reason being that* $B(s)$ *is always convergent for* $s = \alpha_n$, *and* $F(s)$ *may become infinite for* $s = -\alpha_n$.

Let us form the entire function of genus one

$$H(s) = \prod_{m=1}^{\infty} \left\{ \left(1 - \frac{s}{\alpha_m}\right) e^{\frac{s}{\alpha_m}} \right\}, \tag{30}$$

and put

$$H_n(s) = \prod_{m=1}^{n} \left\{ \left(1 - \frac{s}{\alpha_m}\right) e^{\frac{s}{\alpha_m}} \right\}, \tag{31}$$

$$b_n(s) = \prod_{m=1}^{n} \left(1 - \frac{s}{\alpha_m}\right), \tag{32}$$

$$f_n(s) = \prod_{m=1}^{n} \frac{1}{1 + \frac{s}{\alpha_m}}, \tag{33}$$

$$d_n(s) = e^{-\lambda_n s}. \tag{34}$$

Then

$$b_n(s) = H_n(s)\, d_n(s); f_n(s) = \frac{d_n(s)}{H_n(-s)}; b_n(s) = H_n(s)\, H_n(-s)\, f_n(s), \tag{35}$$

and

$$B(s) = \sum a_n b_n(s);\ F(s) = \sum a_n f_n(s);\ D(s) = \sum a_n d_n(s). \tag{36}$$

The proof is based upon Lemma 3 and depends upon the convergence of the following four series

$$\sum | H_n(s) - H_{n+1}(s) |, \tag{37}$$

$$\sum \left| \frac{1}{H_n(s)} - \frac{1}{H_{n+1}(s)} \right|, \tag{38}$$

$$\sum | H_n(s)\, H_n(-s) - H_{n+1}(s)\, H_{n+1}(-s) |, \tag{39}$$

$$\sum \left| \frac{1}{H_n(s)\, H_n(-s)} - \frac{1}{H_{n+1}(s)\, H_{n+1}(-s)} \right|. \tag{40}$$

It is enough to show the convergence of the first series, the proof for the other three is similar. We have

$$| H_n(s) - H_{n+1}(s) | = | H_n(s) | \left| 1 - \left(1 - \frac{s}{\alpha_{n+1}}\right) e^{\frac{s}{\alpha_{n+1}}} \right| \leq | H_n(s) | \frac{K_n | s |^2}{| \alpha_{n+1} |^2},$$

where K_n is uniformly bounded in every finite region of the s-plane. The factor $H_n(s)$ is also bounded in such a region since it approaches $H(s)$ uniformly. According to our assumptions the series $\sum \frac{1}{|a_n|^2}$ is convergent which completes the proof.

Now let us prove that the convergence of $D(s)$ implies that of $B(s)$ and *vice versa*. Lemma 3 states the convergence of the series $\sum b_n c_n$ provided the series $\sum b_n$ and $\sum |c_{n+1} - c_n|$ are convergent. Let us first put

$$b_n = a_n d_n(s), \qquad c_n = H_n(s).$$

The series (37) is convergent, hence the convergence of $D(s)$ implies that of $B(s)$. Secondly put

$$b_n = a_n b_n(s), \qquad c_n = \frac{1}{H_n(s)}.$$

The series (38) being convergent, provided $s \neq a_n$, we conclude that the convergence of $B(s)$ implies that of $D(s)$.

The part of the theorem which is concerned with $D(s)$ and $F(s)$ has already been proved by Pincherle.* The proof given above follows the same lines as Pincherle's proof.

In view of theorem 9 we conclude that the theorems 1—6 hold, *mutatis mutandis*, for factorial und binomial series.

* Loc. cit. pp. 384–385. Cf. I. Bendixson, Sur une extension à l'infini de la formule d'interpolation de Gauss, Acta Math., vol. 9 (1886), pp. 1–34, for the idea of using the entire function $H(s)$ in this kind of an investigation.

PRINCETON UNIVERSITY.

ON THE TYPES OF MONOIDAL INVOLUTIONS.

By Virgil Snyder.

A monoidal space transformation is defined as one in which the planes of space are transformed into surfaces (monoids) of order n with a common $(n-1)$-fold point. The theory of these transformations has been outlined by Montesano,* who also showed that the earlier classification of Martinetti† was not complete. On account of the interest these transformations have in the determination of possible irrational involutions, and by the existence of fundamental elements of a kind not found in other types‡ it seems desirable to obtain the forms of the equations of these involutions, and to discuss some cases of particular interest. Montesano showed§ that the bundle of lines through the vertex is transformed into a congruence of lines of order one, which must be either

(*a*) the same bundle,

(*b*) a different bundle,

(*c*) lines meeting a line and a curve of order m having $m-1$ points on the line, or

(*d*) the bisecants of a cubic curve.

Among these are included all the types in which a line joining a pair of conjugate points contains an infinite number of such pairs, discussed by De Paolis.‖

All of these types belonging to a congruence of lines can be transformed into the central monoidal type¶ hence only this form need be considered.

1. Let $O = (0, 0, 0, 1)$ be the vertex of the monoid. Since the bundle of lines (O) is transformed into itself, and is involutorial, the bundle must

* D. Montesano, *Su le trasformazioni involutorie monoidali*, Rend. del R. Ist. Lombardo, ser. 2, vol. 21 (1888), pp. 579–594; 684–687.

† L. Martinetti, *Sopra una classe di trasformazioni involutorie dello spazio*, Rend. del R. Ist. Lombardo, ser. 2, vol. 18 (1885), pp. 132–141.

‡ H. Hudson, *On fundamental points of Cremona space transformations*, Annali di Matematica, ser. 3, vol. 19 (1912), pp. 45–54; A. Tummarello, *Su i sistemi omaloidici costituiti da monoidi aventi contatto di second'ordine nel vertice*, Note e Memorie di Matematica, vol. 1 (1921), pp. 289–298; M. Beloch, *Sulle trasformazioni birazionali nello spazio*, Annali di Matematica, ser. 3, vol. 16 (1909), pp. 27–68.

§ l. c.

‖ R. De Paolis, *Alcune particolari trasformazioni involutorie dello spazio*, Nota I, Rend. dei Lincei, ser. 4, vol. 1 (1885), pp. 735–742.

¶ F. R. Sharpe and V. Snyder, *Certain types of involutorial space transformations*, Trans. Amer. Math. Soc., vol. 20 (1919), pp. 185–202. See p. 200 ff.

be transformed by one of the five types of the ternary involution* namely, the identity, the harmonic homology (H), the Geiser (G), the Bertini (B), and the Jonquières (J).

In the first case, the conjugate of a point P is on the line OP, and is the harmonic conjugate of as to the two residual points of intersection of OP with a fixed surface of order n, having an $(n-2)$-fold point at O. If the equation of this surface is

$$u_{n-2}\, x_4^2 + 2\, u_{n-1}\, x_4 + u_n = 0,$$

u_i being ternary in x_1, x_2, x_3 of degree i, the equations of the transformation are

$$rx_i' = x_i,\ i = 1, 2, 3, \qquad rx_4' = \frac{-(u_{n-1}\, x_4 + u_n)}{u_{n-2}\, x_4 + u_{n-1}}.$$

The basis element is the curve g of order $n(n-1)$ common to the two monoids $u_{n-1}\, x_4 + u_n = 0$, $u_{n-2}\, x_4 + u_{n-1} = 0$. Its image is the cone K of order $2(n-1)$ which projects it from O. The cone touches the given surface along g. The image of O is the tangent cone $u_{n-2} = 0$. The conjugate of any line in space is a plane curve lying in the plane determined by the given line and O. Every plane through O is invariant and contains a perspective (J) involution from O.

If u_{n-2} is identically zero, the given surface is itself a monoid, and P' is the harmonic conjugate of P as to O and the residual point of intersection with OP. In this case the image of O is the cone $u_{n-1} = 0$, which is common to all the monoids of the system. The residual curve of intersection of the monoids of the web is now replaced by the $n(n-1)$ lines a_i common to the two cones $u_{n-1} = 0$, $u_n = 0$; these basis lines are fundamental lines of the second kind. The image of a point on any one of them is the whole line passing through it. If the lines are all distinct, the genus of the transformation is the maximum, but it is lowered by multiple basis lines. If a_i is a basis line of multiplicity k then any point of a_i has for image the whole line a_i counted k-fold. The possibility of this phenomenon was recently pointed out by Montesano.†

2. The second species, MH is of the form

$$rx_1' = x_2, \quad rx_2' = x_1, \quad rx_3' = x_3, \quad rx_4' = \frac{-(u_{n-1}\, x_4 + u_n)}{u_{n-2}\, x_4 + u_{n-1}},$$

in which each u_i is ternary, and is symmetric in x_1, x_2.

* E. Bertini, *Ricerche sulle trasformazioni univoche involutorie nel piano*, Annali di Mat., ser. 2, vol. 8 (1877), pp. 244–286.

† D. Montesano, *Sulla teoria generale delle corrispondenze birazionali dello spazio*, Rend. dei Lincei, ser. 5, vol. 27 (1918), pp. 396–400, 438–441; ser. 5, vol. 30 (1921), pp. 447–451.

Lines joining corresponding points belong to the special linear complex $p_{13} + p_{23} = 0$. The invariant points lie on the curve

$$u_{n-2}\, x_4^2 + 2\, u_{n-1}\, x_4 + u_n = 0, \qquad x_1 + x_2 = 0.$$

The 11 forms enumerated by Martinetti* can now be found immediately from the preceding equations, by particularizing the u_i appropriately.

3. Any type is expressible in the form

$$x_i' = \varphi_i, \qquad i = 1, 2, 3, \qquad x_4' = \frac{x_4 \varphi u + v}{w x_4 - u x},$$

in which the φ_i define an involutorial Cremona net of cones with vertices at 0. φ is a linear function of the φ_i and x is the same linear function of the x_i. The expressions u, v, w are algebraic polynomials in functions k_i that are invariant under the ternary involution. These will be discussed in each type separately.

4. Type MG. Let $\varphi_i = 0$ be cones of order 8, having 7 triple basis lines h_i. The functions k_i are now cubics, each $k_i = 0$ being a cubic cone passing simply through each of the seven basis lines h_i. Now if v is of order n, then we take u and w each of order $n-3$ in (k). The transformation is of order $3n$ and has the symbol

$$s_1 \sim s_{3n}\,(O^{3n-1}\, h_1^n \cdots h_7^n\, C),$$

in which C is the curve of intersection of the two monoids

$$x_4 \varphi u + v = 0, \qquad x_4 w - u x = 0.$$

It is of order $3n(3n-8) - 7n(n-3) = n(2n-3)$, and is projected from O by the cone $u^2 \varphi x_4 + v w = 0$ of order $3(2n-3)$; hence O is a point of multiplicity $(n-3)(2n-3)$ on C. The image of O is the monoid $x_4 w - u x = 0$, $M_{3n-8}\,(O^{3n-7}\, h_i^{n-3}\, C)$. The curve of invariant points is the intersection of the surfaces

$$\frac{x_4}{x_i} = \frac{x_4 \varphi u + v}{(w x - u x)\, \varphi_1}, \qquad R_6 = 0,$$

where $R_6 = 0$ is the invariant sextic cone $R_6(h_1^2, \ldots, h_7^2)$ of (G). Since every $x_i = 0$ meets its conjugate $\varphi_i = 0$ in six lines of $R_6 = 0$ and in

* l. c.

a pair of conjugate rays in (G), it follows that the invariant curve is of order $6(3n+1)-7\cdot 2\cdot n-6 = 4n$.

An interesting particular case arises when $n=3$. Now u is a constant. Suppose v and w are identically zero. The image of O is the plane $x=0$ passing through it, and the transformation is of order 9. The curve C consists of six coplanar lines of $R_6=0$, a pair of lines through O, conjugate in (G), and the line m; $x=0$, $x_4=0$. The cone $\varphi=0$ is the tangent cone to all the monoids of the system at O. Every plane through m is transformed into itself, hence every line joining a pair of conjugate points meets m. The image of m is the cone $\varphi=0$. The image of a line meeting m consists of a generator of $\varphi=0$, image of the point of intersection, and of a plane curve c_8 of order 8 in the plane determined by the given line and m; this curve is the section of the cone which is the image of the plane which joins the given line to O. The line meets its conjugate c_8 in six points on $R_6=0$, and in a pair of conjugate points; that is, every line meeting m contains just one pair of conjugate points. The involution is therefore rational, since every line of a special linear complex contains a single pair of conjugate points.

If the plane $x=0$ passes through a line h_i, the common tangent cone $\varphi=0$ is composite, consisting of a cubic cone $k_i=0$ passing through all the basis lines h, and having h_i for double generator, and of an elliptic quintic.

If $x=0$ passes through two basis lines, the composite image cone consists of two such cubic cones, and of the quadric cone through the remaining five. Numerous subcases exist when the lines h_i of the Geiser net have restricted positions, but these can all be interpreted directly, since the resulting forms of the Geiser cones have been determined.*

5. ***MB.*** The cones of the Bertini transformation are of order 17, and have 8 basis lines h_i each to multiplicity 6. The pencil of cubic cones through these lines is transformed into itself, leaving two cones of the pencil invariant, k_1 and k_2. If v is of order n in k_1 and k_2, while u and w are each of order $n-6$ the resulting space transformation is of order $3n$. The image of O is the monoid $x_4 w_{n-6}(k) - u_{n-6}(k)x = 0$ of order $3n-17$.

The residual basis curve C is of order $3n(3n-17) - 8n(n-6) = n(n-3)$, and is projected from O by the cone of order $6(n-3)$, hence O is of multiplicity $(n-6)(n-3)$ on $C_{n(n-3)}$.

The curve of invariant points is the intersection of the surfaces

$$\frac{x_4}{x_i} = \frac{x_4 u\varphi + v}{-(x_4 w - ux)\varphi_i}, \qquad R_9(h_1^3, \ldots, h_8^3) = 0,$$

* V. Snyder, *Conjugate line congruences contained in a bundle of quadric surfaces*, Trans. Amer. Math. Soc., vol. 11 (1910), pp. 371–387.

in which $R_9 = 0$ is the cone of invariant lines of the Bertini ternary transformation. The curve is of order $3n$.

For values of $n \geqq 7$, we have the web of sextic cones invariant under (B),* and for $n = 9$ the R_9 may also be employed in u, v, w; the form of the result, as a function of n, is not changed.

If now $n = 6$, and v, w are each taken identically zero, the result is a transformation of order 18; the image of O is the plane $x = 0$, and every line joining a pair of conjugate points meets the fundamental line m: $x = 0$, $x_4 = 0$. The conjugate of an arbitrary line is a rational curve of order 18, which passes simply through O. If the line meets m, then the image of the point of intersection is a component of the conjugate; the proper image is a plane curve of order 17 in the plane determined by m and the given line. It is a plane (B) transformation. The line meets $R_9 = 0$ in 9 points, which also lie on the conjugate c_{17}; hence it meets c_{17} in eight other points, which are four pairs of conjugates.

THEOREM: *Every line joining a pair of conjugate points in the special MB transformation contains three other pairs.*

It probably includes irrational types.

The cone $\varphi = 0$ is common tangent cone to all the monoids of the web. It is composite when $x = 0$ contains one or more basis lines h_i. When the lines have restricted positions, this case can be reduced to a type included among the preceding ones.†

18. Type MJ. The equations of the plane perspective Jonquières transformation of order n have the form

$$rx_1' = x_1(u_{n-2}x_3 + u_{n-1}), \quad rx_2' = x_2(u_{n-2}x_3 + u_{n-1}), \quad rx_3' = -(u_{n-1}x_3 + u_n),$$

in which each u_i is a binary form of order i in x_1, x_2. The associated space involution is defined by the preceding equations and

$$rx_4' = \frac{x_4[(c_1x_1 + c_2x_2)(u_{n-2}x_3 + u_{n-1}) - c_3(u_{n-1}x_3 + u_n)]U + V}{W - U[c_1x_1 + c_2x_2 + c_3x_3]},$$

in which U, V, W are functions of x_1, x_2, x_3 invariant under (J). The fundamental lines are k: $x_1 = 0$, $x_2 = 0$, and $2(n-1)$ simple lines p_i, and the invariant cones have the symbol $k^r p_i^s$, wherein $r > s(n-1)$.

Every plane through the line k goes into itself in (J), hence:

* V. Snyder, *The involutorial birational transformation of the plane, of order* 17, Amer. Jour. Math., vol. 23 (1911), pp. 327–336.

† V. Snyder, l. c., Amer. Jour., vol. 23 (1911). See p. 334.

THEOREM: *Every monoidal involution of the form* MJ *belongs to a special linear complex, and the axis of the complex passes through the common vertex of the monoids.**

19. In particular, if V, W are identically zero and U a constant, not zero, the involution belongs to a linear congruence having the directrices k and m. Every line of this congruence is invariant under this particular MJ, and each contains an infinite number of pairs of conjugate points. The cone $R_n = 0$; $x_4^2 u_{n-2} + 2x_4 u_{n-1} + u_n = 0$ is composed of invariant points, and contains the line k to multiplicity $n-2$. The image of any point P is its harmonic conjugate, on the line of the congruence through P, as to the two residual points of intersection with the cone $R_n = 0$. This case can be reduced to a perspective monoidal form.

20. A still more particular case arises when $c_3 = 0$ in the equations of the transformation, which now reduce to the form

$$\begin{aligned} rx_1' &= x_1(u_{n-2}x_3 + u_{n-1}), & rx_2' &= x_2(u_{n-2}x_3 + u_{n-1}), \\ rx_3' &= -(u_{n-1}x_3 + u_n), & rx_4' &= x_4(u_{n-2}x_3 + u_{n-1}), \end{aligned}$$

that is, the image of a plane is monoidal as to $(0, 0, 1, 0)$ as well as to $(0, 0, 0, 1)$. A line joining a pair of conjugate points now belongs to the bundle $(0, 0, 1, 0)$.

From these examples we see that if an irrational involution can be reduced to the monoidal form it is either MB or MJ.

* The involution belonging to the cubic variety of four way space belongs in this category. In its simplest form it is of order six and a line joining a pair of conjugate points also contains a second pair. This involution is very probably irrational.

CORNELL UNIVERSITY,
June, 23.

AN INEQUALITY FOR THE ROOTS OF AN ALGEBRAIC EQUATION.

By J. L. Walsh.

This note gives an inequality for the roots of an algebraic equation which can be proved from the following theorem:

If the roots of the equation

$$x^m + c_1 x^{m-1} + \cdots + c_m = 0$$

lie on or within a circle C whose radius is r and whose center is the point $x = \alpha$, then all the roots of the equation

$$x^m + c_1 x^{m-1} + \cdots + c_m = M \tag{1}$$

*lie on or within the m circles whose centers are the m points $\alpha + M^{\frac{1}{m}}$ and whose common radius is r.**

In the application of this theorem we shall use merely the fact that the roots of (1) lie on or within the circle whose center is α and radius $r + |M^{\frac{1}{m}}|$.

All roots of the equation

$$x^2 + a_1 x = 0$$

lie on or within the circle whose center is the origin and radius $|a_1|$. Then all roots of

$$x^2 + a_1 x + a_2 = 0$$

lie on or within the circle whose center is the origin and radius $|a_1| + \sqrt{|a_2|}$. Hence all roots of

$$x^3 + a_1 x^2 + a_2 x = 0$$

lie on or within the same circle, so all roots of

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

* Walsh, Trans. Amer. Math. Soc., vol. 24 (1922).

lie on or within the circle whose center is the origin and radius $|a_1| + \sqrt{|a_2|} + \sqrt[3]{|a_3|}$.

Continuation of this reasoning shows that all roots of the equation

$$x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_n = 0 \tag{3}$$

lie on or within the circle whose center is the origin and whose radius is

$$|a_1| + \sqrt{|a_2|} + \sqrt[3]{|a_3|} + \cdots + \sqrt[n]{|a_n|}. \tag{4}$$

This upper limit (4) for the moduli of the roots of (3) is actually attained when all but one of the coefficients a_i vanish, and is thus attained not merely for a single type of equation (3) but for n types of such equations. The upper limit (4) is particularly useful when all but one of the coefficients a_i are small when compared with the remaining coefficient.

If we use the fact that all the roots of equation (2) lie on or within the circle whose center is $-a_1/2$ and radius $|a_1/2|$, we find that all the roots of (3) lie on or within the circle whose center is $-a_1/2$ and whose radius is

$$\left|\frac{a_1}{2}\right| + \sqrt{|a_2|} + \sqrt[3]{|a_3|} + \cdots + \sqrt[n]{|a_n|}.$$

DIFFERENTIAL EQUATIONS FROM THE GROUP STANDPOINT.

By L. E. Dickson.

Introduction.

The various classic devices for the integration of differential equations may be explained simply from a single standpoint—that of infinitesimal transformations leaving the equations invariant. What is still more important than this unification of diverse known methods, infinitesimal transformations furnish us a new tool, likely to succeed when the ordinary methods fail, since they enable us to take into account vital information ignored by the ordinary methods. In fact, the new method does not confine attention to the differential equation and ignore the data of the problem of which the equation is an analytic formulation, but makes use of the data itself in order to obtain one or more infinitesimal transformations leaving the equation invariant. Accordingly the new method is most readily applied successfully to differential equations arising in geometry or mechanics. Why bother with a dead equation whose origin is unknown or has been concealed?

Although no previous acquaintance with differential equations is presupposed, the paper is not proposed as a substitute for the usual introductory course, but rather to provide a satisfactory review *ab initio* and at the same time to present the unifying and effective method based on groups.

The important topic of differential invariants is given considerable attention at appropriate places throughout the paper. Application is made in § 54 to the congruence of plane curves and their intrinsic equations.

Finally, we obtain in § 55 a complete set of functionally independent covariants and invariants of the general binary form and deduce the facts that every polynomial invariant of the binary quadratic or cubic form is a polynomial function of its discriminant, while every polynomial invariant of a quartic form is a polynomial function of two specified invariants. This method of attack provides an easy introduction to the complicated algebraic theory of invariants as well as the relation between that subject and the topic of functionally independent invariants.

The writer is greatly indebted to the founder of the theory of continuous groups, Sophus Lie, whose lectures he attended in 1896. Numerous valuable suggestions on the manuscript were received from Professor Bliss. It has been used in classes by Dr. Barnett and the writer.

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CHAPTER I.

One-parameter groups of transformations.

The object of this chapter is to define and illustrate the concepts transformation and group of transformations whose equations involve a single parameter. Each such group is generated by an infinitesimal transformation. The latter will prove to be more convenient for the subsequent applications than the group.

1. **Translations.** In analytic geometry we interpreted the pair of equations

$$(1) \qquad x_1 = x + 4, \qquad y_1 = y + 3$$

as relations between the coördinates x, y of any point P referred to one pair of rectangular axes Ox, Oy, and the coördinates x_1, y_1 of the same point P referred to another pair of rectangular axes O_1x_1, O_1y_1, parallel to the former axes respectively, and such that O has the coördinates (4,3) with respect to the axes O_1x_1, O_1y_1.

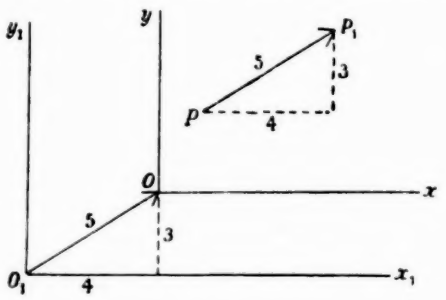

Fig. 1.

On a sheet of transparent paper covering Fig. 1, draw traces of the lines O_1x_1 and O_1y_1 and of the point P having the coördinates x_1, y_1 referred to those axes. Move the transparent sheet without rotation (so that O_1x_1 remains parallel to itself during the motion) until the trace of O_1 covers O and hence the trace of O_1x_1 covers Ox. Under this *translation* of the transparent sheet, the trace of P moves to the point P_1 having the coördinates x_1, y_1 referred to the axes Ox, Oy.

This leads us to the following new interpretation of equations (1). We ignore henceforth the axes O_1x_1 and O_1y_1 and regard x, y as the coördinates of one point P and x_1, y_1 as the coördinates of a new point P_1, each referred to the same axes Ox, Oy. In view of equations (1), the line PP_1 is parallel to O_1O and of the same length 5 as O_1O. Under our new interpretation, formulas (1) accomplish a *translation* of all points of the plane through a distance 5 in a direction parallel to O_1O.

2. **Rotations.** If x, y and x_1, y_1 are the coördinates of the same point P referred to two pairs of rectangular axes Ox, Oy and Ox_1, Oy_1, respectively, such that t is the positive angle measured counter-clockwise from Ox_1 to Ox, it is proved in analytic geometry that

$$(2) \qquad x_1 = x\cos t - y\sin t, \qquad y_1 = x\sin t + y\cos t.$$

On a sheet of transparent paper covering Fig. 2, draw traces of the lines Ox_1, Oy_1 and of the point P having the coördinates x_1, y_1 referred to those axes. Rotate the transparent sheet counter-clockwise about O through angle t, so that the trace of Ox_1 will now cover Ox. Then the trace of $P:(x, y)$ is rotated to the position P_1 shown in Fig. 2.

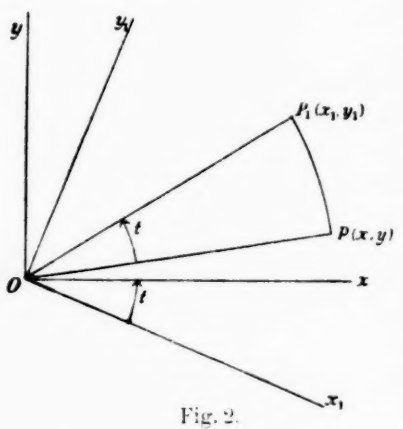

Fig. 2.

This leads us to the following new interpretation. When both points (x, y) and (x_1, y_1) are referred to the same rectangular axes Ox, Oy, equations (2) define a *rotation* of all points of the plane about the origin O counter-clockwise through angle t.

3. **Transformations.** Generalizing from the translation (1) and the rotation (2), we shall say that any pair of equations of the type

$$x_1 = g(x, y), \qquad y_1 = h(x, y)$$

defines a *transformation* of all points of the plane, provided g and h are independent functions of x, y. It is again understood that the arbitrary initial point (x, y) and its transformed point (x_1, y_1) are both referred to the same pair of rectangular axes.

4. **Product of two translations.** If a is a positive constant, the pair of equations

$$T_a: \qquad x_1 = x + a, \qquad y_1 = y$$

represents a translation through a distance a parallel to the x-axis and toward its positive direction. Let

$$T_b: \qquad x_1 = x + b, \qquad y_1 = y$$

represent another such translation; it carries the point (x, y) to the point $(x + b, y)$. Since x and y are arbitrary and hence may be assigned the values x_1 and y_1, we see that T_b carries the point (x_1, y_1) to the point $(x_1 + b, y_1)$, which will be designated also by (x_2, y_2). We may therefore express T_b in the form

$$T_b: \qquad x_2 = x_1 + b, \qquad y_2 = y_1.$$

Since T_a carries the point (x, y) to the point (x_1, y_1), which T_b carries to the point (x_2, y_2), the combined effect of the two translations acting in succession is to carry the point (x, y) to the point (x_2, y_2) such that

$$x_2 = x_1 + b = x + a + b, \qquad y_2 = y_1 = y.$$

This replacement of the point (x, y) by the point (x_2, y_2) or $(x+a+b, y)$ is also accomplished by the single translation

$$T_{a+b}: \qquad x_1 = x + (a+b), \qquad y_1 = y.$$

This latter translation is called the *product* of T_a and T_b, taken in this order, and is designated by $T_a T_b$. The word "product" is here a technical term having the sense of "compound" or "resultant". Our conclusion,

$$T_a T_b = T_{a+b},$$

is also evident from mechanical considerations. The effect of two successive translations through the distances a and b, each in a direction parallel to the positive x-axis, is the same as that produced by the translation through the distance $a+b$ parallel to the positive x-axis.

5. **Inverse of a translation, identity transformation.** In the translation T_b we now permit b to take the negative value $-a$. Then the pair of equations

$$T_{-a}: \qquad x_1 = x - a, \qquad y_1 = y$$

represents a translation through a distance a parallel to the x-axis, but toward its negative direction. The effect of T_{-a} is therefore just the reverse of that of T_a; it undoes what was done by T_a, or effects the "return trip". For, T_a translates the point (x, y) to $(x+a, y)$, while T_{-a} translates the latter point to $(x+a-a, y)$, which is the initial point (x, y). For these reasons we shall call T_{-a} the *inverse* of T_a and designate it by T_a^{-1}.

The product $T_a T_a^{-1}$ is $x_1 = x$, $y_1 = y$, which is called the *identity transformation* and designated by I, since the transformed point (x_1, y_1) under I is always identical with the initial point. Also, $T_{-a} T_a = I$. Thus

$$T_a^{-1} = T_{-a}, \qquad T_a T_a^{-1} = T_a^{-1} T_a = I.$$

Similar ideas are involved in the definition of the inverse trigonometric functions. For example, if the sine of angle A has the value r, then A is called the inverse sine of r, and we write $A = \sin^{-1} r$. Thus $\sin(\sin^{-1} r) = r$.

6. **Groups of transformations.** The set of translations

$$T_a: \qquad x_1 = x + a, \qquad y_1 = y,$$

obtained when a ranges over all *integral* values (i. e., the positive and negative whole numbers and zero), is said to constitute a *group* since it has the following two properties.

(i) The product of any two (distinct or coincident) transformations of the set is itself one of the transformations of the set.

(ii) The inverse of any transformation of the set is itself a transformation of the set.

For, if a and b are any two integers (distinct or equal),

$$T_a T_b = T_c, \qquad c = a + b, \qquad T_a^{-1} = T_{-a}$$

and c and $-a$ are also integers. This group is called *discontinuous* since the parameter a has the discontinuous range of all integral values. Discontinuous groups are useful in the theory of numbers, the theory of elliptic modular functions, and Galois' theory of equations. They will be excluded in what follows.

But the set of translations T_a in which a ranges over all real numbers forms a *continuous one-parameter* group since the parameter a has a continuous range of values and since the formulas for T_{a_1} may be converted into the formulas for T_{a_2} by continuous variation of the parameter from a_1 to a_2.

The second condition of this definition of a continuous group is not satisfied for the *mixed group* composed of all the T_a together with the transformations

$$R_a: \qquad x_1 = x + a, \qquad y_1 = -y,$$

where again a ranges over all real numbers. We have a group since

$$R_a R_b = T_{a-b}, \qquad R_a T_b = T_b R_a = R_{a+b}, \qquad R_a^{-1} = R_{-a}.$$

We shall not employ mixed groups in this paper.

The set of rotations (2) in which t ranges over all real numbers is a continuous one-parameter group. For, the product of rotation (2) by the rotation

$$x_2 = x_1 \cos t' - y_1 \sin t', \qquad y_2 = x_1 \sin t' + y_1 \cos t'$$

is readily verified to be the rotation about the origin counter-clockwise through angle $t + t'$, while the inverse of (2) is the rotation through the angle $-t$.

Again, all similarity transformations

$$x_1 = ax, \qquad y_1 = ay$$

form a continuous one-parameter group. The same is true of

$$\begin{cases} x_1 = ax, \\ y_1 = y. \end{cases} \quad \begin{cases} x_1 = ax, \\ y_1 = \frac{1}{a} y. \end{cases} \quad \begin{cases} x_1 = x + t, \\ y_1 = \frac{xy - t}{x + t}. \end{cases} \quad \begin{cases} x_1 = \frac{x}{1 - ax}, \\ y_1 = \frac{y}{1 - ax}. \end{cases}$$

The next discussion shows how to find as many continuous one-parameter groups as we please, in fact, one group from each pair of functions $\xi(x, y)$, $\eta(x, y)$, which are continuous in a certain region of the xy-plane.

7. **Groups found by integrating differential equations.** We employ the system of differential equations

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \qquad \frac{dy_1}{dt} = \eta(x_1, y_1). \tag{3}$$

These imply the equation in two variables

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)}, \tag{4}$$

which is known* to possess one and only one integral curve $v(x_1, y_1) = c$ passing through any given point (x, y) and defined for all points (x_1, y_1) of a certain region containing (x, y). Evidently $c = v(x, y)$.

In order to integrate the system (3), we solve† $v(x_1, y_1) = c$ for y_1 in terms of x_1 and c, substitute in the first equation (3), and integrate. Let the result be $w(x_1, c) - t = c'$, where c' is an arbitrary constant. Replacing c by its value $v(x_1, y_1)$, we obtain a result of the form $u(x_1, y_1) - t = c'$. Without loss of generality we may assume that $t = 0$ when $x_1 = x$, $y_1 = y$. In view of these initial values, we have

$$u(x_1, y_1) = u(x, y) + t, \qquad v(x_1, y_1) = v(x, y). \tag{5}$$

Let the solved form of these equations (5) be

$$x_1 = \Phi(x, y, t), \qquad y_1 = \psi(x, y, t), \tag{6}$$

* Picard, Traité d'Analyse, II, 1893, pp. 292, 301, 304; III, 1896, p. 88. Bliss, Princeton Colloquium Lectures, 1913, p. 86.

† In case v is independent of y_1, we interchange the rôles of x_1, y_1 in what follows.

in which the functions are defined for values of t which are sufficiently small numerically.

We shall prove that these transformations (6) form a group by verifying that they have the two properties listed in § 6. The product of (6) by a second transformation

$$x_2 = \Phi(x_1, y_1, t'), \qquad y_2 = \psi(x_1, y_1, t') \tag{7}$$

of the same set is the following transformation:

$$x_2 = \Phi(x, y, t + t'), \qquad y_2 = \psi(x, y, t + t'). \tag{8}$$

To simplify the proof, we return from (6) to the equivalent equations (5), and likewise from (7) to the equivalent equations

$$u(x_2, y_2) = u(x_1, y_1) + t', \qquad v(x_2, y_2) = v(x_1, y_1), \tag{9}$$

whose solved form is (7) for the same reason that (6) is the solved form of (5). Eliminating x_1, y_1 between (5) and (9), we evidently get

$$u(x_2, y_2) = u(x, y) + t + t', \qquad v(x_2, y_2) = v(x, y), \tag{10}$$

whose solved form is (8). Hence the product of two transformations (6) whose parameters t and t' are sufficiently small numerically is a transformation (6) with the parameter $t + t'$.

Since (10) is the identity transformation if $t + t' = 0$, the inverse of (6) is derived from (6) by replacing t by $-t$.

If we take $X = u(x, y)$ and $Y = v(x, y)$ as new variables, we obtain from (5) the translation

$$X_1 = X + t, \qquad Y_1 = Y. \tag{11}$$

THEOREM. *When the parameter t is restricted to values sufficiently small numerically, the solutions* (6) *of the system of differential equations* (3) *form a continuous one-parameter group, which can be reduced to the group of translations* (11) *by the introduction of new variables.*

For example, if $\xi = -y_1$, $\eta = x_1$, (4) becomes $x_1 dx_1 + y_1 dy_1 = 0$, which has the integral $v = \sqrt{x_1^2 + y_1^2} = c$. The second equation (3) may be written in the form

$$\frac{dy_1}{\sqrt{c^2 - y_1^2}} = dt, \qquad \sin^{-1}\frac{y_1}{c} = t + c', \qquad y_1 = c \sin(t + c').$$

Then

$$x_1 = c\cos(t + c'), \qquad u = \tan^{-1}\frac{y_1}{x_1} = t + c',$$

$$\tan^{-1}\frac{y_1}{x_1} = \tan^{-1}\frac{y}{x} + t, \qquad \sqrt{x_1^2 + y_1^2} = \sqrt{x^2 + y^2}. \tag{5'}$$

Employing polar coördinates θ, ρ, we get

$$\theta_1 = \theta + t, \qquad \rho_1 = \rho,$$

which are equations of type (11). This group of rotations may be given the form (2) by solving equations (5′) for x_1, y_1.

8. **Infinitesimal transformations.** In view of the equations (6) of a group, any function $f(x_1, y_1)$ of x_1 and y_1 is a function of x, y, and t. We have

$$\frac{df(x_1, y_1)}{dt} = \frac{\partial f(x_1, y_1)}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f(x_1, y_1)}{\partial y_1}\frac{dy_1}{dt}.$$

Making use of (3), we get

$$\frac{df(x_1, y_1)}{dt} = \frac{\partial f(x_1, y_1)}{\partial x_1}\xi(x_1, y_1) + \frac{\partial f(x_1, y_1)}{\partial y_1}\eta(x_1, y_1). \tag{12}$$

When $t = 0$, x_1 becomes x, and y_1 becomes y. Hence

$$\left[\frac{df(x_1, y_1)}{dt}\right]_{t=0} = Uf(x, y) = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}. \tag{13}$$

We deduced (12) from the pair of equations (3). Conversely, (12) implies (3), as seen by taking $f(x_1, y_1)$ to be x_1 and y_1 in turn. Hence (12) is a convenient synthesis of the pair of equations (3). By the integration of (3) we obtained the equations (6) of a group. Accordingly we shall say that equations (3), or their synthesis (12), generate the equations (6) of the group. Dropping the subscripts 1 in (12), we obtain the expression denoted by Uf in (13). With Sophus Lie, we shall speak of Uf as the (symbol of the) *infinitesimal transformation which generates the equations* (6) *of the transformations of the group.* In brief, we shall say that Uf generates the group.

Not only does the symbol Uf give the pair of functions $\xi(x, y)$, $\eta(x, y)$ and hence the differential equations (3) whose integration yields the equations (6) of the group, but conversely the equations (6) determine Uf since

$$\left[\frac{dx_1}{dt}\right]_{t=0} \equiv \left[\frac{\partial \Phi(x, y, t)}{\partial t}\right]_{t=0} \equiv \xi(x, y),$$
$$\left[\frac{dy_1}{dt}\right]_{t=0} \equiv \left[\frac{\partial \psi(x, y, t)}{\partial t}\right]_{t=0} \equiv \eta(x, y), \tag{14}$$

of which (13) is the synthesis.

For $\xi \equiv -y$, $\eta \equiv x$, the infinitesimal transformation is

$$Uf \equiv -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

By the example in § 7, Uf generates the group of all rotations (2) about the origin. Starting with that group, we readily find the preceding infinitesimal transformation by differentiation. For, by (2) and (14),

$$\frac{dx_1}{dt} = -x\sin t - y\cos t = -y_1, \qquad \frac{dy_1}{dt} = x_1;\ \xi \equiv -y,\ \eta \equiv x.$$

By Maclaurin's theorem any function $f_1 \equiv f(x_1, y_1)$, which is regular at the point (x_1, y_1), may be expanded into a series in t for t sufficiently small numerically:

$$f_1 = [f_1]_{t=0} + t\left[\frac{df_1}{dt}\right]_{t=0} + \frac{t^2}{1\cdot 2}\left[\frac{d^2f_1}{dt^2}\right]_{t=0} + \cdots.$$

Denote the second member of (12) by U_1f_1, which therefore becomes Uf (x, y) in (13) when $t = 0$. Hence $df_1/dt = U_1f_1$. Since U_1f_1 is itself a function of x_1, y_1, the same formula gives

$$\frac{d^2f_1}{dt^2} = \frac{d(U_1f_1)}{dt} = U_1(U_1f_1), \qquad \frac{d^3f_1}{dt^3} = U_1[U_1(U_1f_1)],$$

etc. Hence

$$(15)\quad f(x_1, y_1) = f + tUf + \frac{t^2}{1\cdot 2}U(Uf) + \frac{t^3}{1\cdot 2\cdot 3}U[U(Uf)] + \cdots,$$

where f denotes $f(x, y)$. In particular, if we take $f(x_1, y_1)$ to be x_1 and y_1 in turn, we get

$$(16)\quad x_1 = x + tUx + \frac{t^2}{2}U(Ux) + \cdots, \qquad y_1 = y + tUy + \cdots,$$

which give the equations, in the form of series, of the transformations of the group generated by Uf. Here t is restricted to the values for which the series converge. We recall also the restriction on t in the equations (6) of the group obtained by integrating differential equations. But these restrictions are unimportant for the applications which depend upon the transformations defined by the values of the parameter in the neighborhood of $t = 0$.

By (13), $Ux = \xi$, $Uy = \eta$. It is a common practice to employ the infinitesimal notation δt for a value of t in the immediate neighborhood of $t = 0$, to neglect higher powers of δt, to write (16) in the form

$$x_1 = x + \xi\,\delta t, \qquad y_1 = y + \eta\,\delta t,$$

and to regard these as the equations of the infinitesimal transformation Uf. This serves to explain the historical origin of the latter term. However, we shall avoid the use of infinitesimal quantities.

Employing the terminology explained here, we may state the theorem of § 7 in the form in which it will be quoted later.

THEOREM. *Every infinitesimal transformation on x and y generates (in the sense of integration) a continuous one-parameter group. After suitably chosen new variables have been introduced, the group becomes the group of translations $x_1 = x + t$, $y_1 = y$, generated by the infinitesimal transformation $\partial f / \partial x$.*

9. **Every one-parameter group is generated by an infinitesimal transformation.** We shall first treat the typical case of the group of similarity transformations

$$S_a: \qquad x_1 = ax, \qquad y_1 = ay.$$

Since $S_a S_b = S_{ab}$, the parameter of the product is the product of the parameters of the component transformations, whereas it was their sum in the case of the group of the transformations (6). Hence we cannot put the transformations S_a into one-to-one correspondence with the transformations of any group of type (6) by taking $a = t$. But this can be done by taking $a = e^t$, where e is the base of natural logarithms, and t takes only real values if we restrict a to positive values. Then

$$x_1 = e^t x, \qquad y_1 = e^t y, \tag{17}$$

and, by (14), $\xi = x$, $\eta = y$. The resulting infinitesimal transformation

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

generates the group (17), whose equations give the solutions of

$$\frac{dx_1}{dt} = x_1, \qquad \frac{dy_1}{dt} = y_1,$$

subject to the initial conditions $x_1 = x$, $y_1 = y$, when $t = 0$.

Using similar ideas, we next treat any one-parameter group of transformations

$$x_1 = g(x, y, a), \qquad y_1 = h(x, y, a). \tag{18}$$

The product of (18) by any other transformation

$$x_2 = g(x_1, y_1, b), \qquad y_2 = h(x_1, y_1, b) \tag{19}$$

of the group must be a third transformation

$$x_2 = g(x, y, c), \qquad y_2 = h(x, y, c) \tag{20}$$

of the group. Hence there exists a function $c = \gamma(a, b)$ such that

$$g(g(x, y, a),\ h(x, y, a),\ b) \equiv g(x, y,\ \gamma(a, b)), \tag{21}$$

identically in x, y, a, b, with a similar identity for h. We differentiate* with respect to b, insert the abbreviations (18), and get

$$\frac{\partial g(x_1, y_1, b)}{\partial b} \equiv \frac{\partial g(x, y, c)}{\partial c} \cdot \frac{\partial \gamma(a, b)}{\partial b}.$$

The group contains the identity transformation I given by the value a_0 of the parameter a. For $b = a_0$, (19) becomes I, and the product of (18) by (19) becomes (18), whence $c = a$. Hence the last identity becomes

$$\left[\frac{\partial g(x_1, y_1, b)}{\partial b}\right]_{b = a_0} \equiv \frac{\partial g(x, y, a)}{\partial a} \left[\frac{\partial \gamma(a, b)}{\partial b}\right]_{b = a_0}. \tag{22}$$

The final factor is unity when $a = a_0$, since (18) is I for $a = a_0$, whence $\gamma(a_0, b) = b$. Since the final factor is not zero identically, and contains the parameter a continuously, we may denote it by $1/w(a)$. Denote the left member of (22) by $\xi(x_1, y_1)$, and the similar expression in h by $\eta(x_1, y_1)$. Then

$$\frac{dx_1}{da} = w(a)\, \xi(x_1, y_1), \qquad \frac{dy_1}{da} = w(a)\, \eta(x_1, y_1). \tag{23}$$

Introduce the new parameter

$$t = \int_{a_0}^{a} w(a)\, da \tag{24}$$

* We assume that the two functions (19) may be differentiated with respect to b.

in place of a. Then

$$dt = w(a)\,da, \qquad \frac{dx_1}{dt} = \frac{dx_1}{da}\cdot\frac{da}{dt};$$

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \qquad \frac{dy_1}{dt} = \eta(x_1, y_1). \tag{25}$$

If $a = a_0$, then $x_1 = x$, $y_1 = y$, $t = 0$, which are the initial values employed in § 7.

THEOREM. *The transformations* (18) *of any one-parameter group whose identity transformation has the parameter* a_0 *are put into one-to-one correspondence, by the introduction of the new parameter* (24), *with the transformations of the group generated by the infinitesimal transformation*

$$\xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y}, \qquad \xi = \left[\frac{\partial g(x, y, a)}{\partial a}\right]_{a=a_0}, \qquad \eta = \left[\frac{\partial h(x, y, a)}{\partial a}\right]_{a=a_0}. \tag{26}$$

These values of ξ, η follow from (22) for $a = a_0$ since $w(a_0) = 1$.

10. **Equivalence of two one-parameter groups.** Two groups

$$x_1 = g(x, y, a), \qquad y_1 = h(x, y, a),$$

$$x_1 = G(x, y, A), \qquad y_1 = H(x, y, A)$$

are called *equivalent* if and only if there exists a one-to-one correspondence $A = \Phi(a)$ such that

$$g(x, y, a) \equiv G(x, y, \Phi(a)), \qquad h(x, y, a) \equiv H(x, y, \Phi(a)),$$

identically in x, y, a. By (23), these imply

$$w(a)\xi(x_1, y_1) \equiv g_a \equiv G_A\,\Phi'(a) \equiv \Phi'(a)\cdot W(A)\,X(x_1, y_1),$$

$$w(a)\eta(x_1, y_1) \equiv h_a \equiv H_A\,\Phi'(a) \equiv \Phi'(a)\cdot W(A)\,Y(x_1, y_1),$$

in which W, X, Y are the functions for the second group defined in the same manner that w, ξ, η were defined for the first group. Taking $a = a_0$, where a_0 gives the identity transformation of the first group, we get

$$\xi(x, y) \equiv k\,X(x, y), \qquad \eta(x, y) \equiv k\,Y(x, y),$$

where k is a constant. Conversely, if the last two identities hold, the groups are equivalent under the correspondence $kt = T$, since from (3) we get

$$\frac{dx_1}{dT} = \frac{dx_1}{kdt} = X(x_1, y_1), \qquad \frac{dy_1}{dT} = \frac{dy_1}{kdt} = Y(x_1, y_1).$$

THEOREM. *Two one-parameter groups are equivalent if and only if the symbols of their infinitesimal transformations differ only by a constant factor.*

11. **Invariant points.** A point may be invariant (left unchanged) by all the transformations of a group. For example, the origin is invariant under all of the transformations

$$x_1 = ax, \qquad y_1 = ay. \tag{27}$$

If a point (x, y) remains unaltered by every defined transformation (6) of a group, so that $x_1 = x$, $y_1 = y$, for every permissible value of t, then

$$\frac{dx_1}{dt}, \qquad \frac{dy_1}{dt}$$

evidently vanish at (x, y). Hence by (3), $\xi(x, y)$ and $\eta(x, y)$ both vanish at (x, y). This also follows from (16), which implies the converse.

THEOREM. *A point (x, y) is invariant under the group generated by the infinitesimal transformation*

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} \tag{28}$$

if and only if ξ and η both vanish at (x, y).

12. **Path curves.** By way of introduction, consider the group G of similarity transformations (27). Any point (x_0, y_0), not the origin, is carried by the various transformations of G into the various points (ax_0, ay_0) whose locus is the straight line $x_0 y = y_0 x$. Moreover, any point $(a'x_0, a'y_0)$ of the latter is carried by G into points of the same line. For these reasons, any straight line through the origin is called a *path curve* of the group G.

Consider any one-parameter group G of transformations

$$T_a: \qquad x_1 = g(x, y, a), \qquad y_1 = h(x, y, a), \tag{29}$$

whose identity transformation is given by the value a_0 of a. Through any point (x_0, y_0), which is not invariant under G, there passes one path curve C_0 whose parametric equations are

$$x = g(x_0, y_0, a), \qquad y = h(x_0, y_0, a), \tag{30}$$

and whose ordinary equation is found by eliminating a. For, if both of these functions were independent of a (so that the elimination of a would be impossible), they would be identical with their values x_0, y_0 for $a = a_0$, whereas the point (x_0, y_0) is not invariant under G. That the curve C_0 passes through (x_0, y_0) is shown by taking $a = a_0$.

If we apply to any point (x, y) on C_0 any transformation T_b of the group G, we obtain a point (x_1, y_1) on C_0. For, we may interpret (30) as a transformation (29) of G which carries (x_0, y_0) to (x, y). Since the product $T_a T_b$ is a transformation T_c of G and carries (x_0, y_0) to (x_1, y_1), we have

$$x_1 = g(x_0, y_0, c), \qquad y_1 = h(x_0, y_0, c),$$

whence the point (x_1, y_1) is on the curve C_0 defined by (30).

If (x, y) and (x', y') are any two points on C_0, so that (x_0, y_0) is carried to them by certain transformations T_a and $T_{a'}$ of G, then (x, y) is carried to (x', y') by the transformation $T_a^{-1} T_{a'}$ of G.

If the curve C_0 has a point (x', y') in common with another path curve of G,

$$C_1: \qquad x = g(x_1, y_1, b), \qquad y = h(x_1, y_1, b),$$

the last result shows that (x', y') is carried to any chosen point P on C_1 by some transformation T of G. But the point (x_0, y_0) is carried to (x', y') by some transformation T' of G. Hence $T'T$ carries (x_0, y_0) to P, so that P lies on C_0 since $T'T$ is a transformation of G. This proves that C_1 coincides with C_0.

THEOREM 1. *Every one-parameter group G has a family of path curves the points on each of which are merely permuted by the transformations of G and such that there is one and only one path curve through each point of the plane not an invariant point.*

Since the equation of any path curve is obtained by eliminating a between the equations (30), which may be interpreted as the equations (29) of the general transformation of the group, and since the functions (29) are the solutions of the system of differential equations (23), it follows (after dropping the subscripts 1) that the path curves are the integral curves of

$$\frac{dx}{\xi(x, y)} = \frac{dy}{\eta(x, y)}, \tag{31}$$

and hence are $v(x, y) = c$ in the notation of § 7. By differentiation,

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = 0. \tag{32}$$

Since dx and dy are proportional to ξ and η by (31),

$$\frac{\partial v}{\partial x} \xi + \frac{\partial v}{\partial y} \eta = 0. \tag{33}$$

Hence $Uv = 0$, where Uf is the infinitesimal transformation (28) which generates the group.

Conversely, if $v(x, y)$ is a solution of $Uf = 0$, (12) shows that $dv(x_1, y_1)/dt = 0$. Thus $v(x_1, y_1)$ is independent of t and hence is equal to its value $v(x, y)$ for $t = 0$. Hence every transformation of the group replaces each point (x, y) of the curve $v(x, y) = c$ by a point (x_1, y_1) on the same curve, which is therefore a path curve.

THEOREM 2. *The path curves of the group generated by an infinitesimal transformation Uf are obtained by equating to an arbitrary constant a solution of $Uf = 0$. The slope of a path curve at (x, y) is η/ξ.*

EXAMPLES. For the infinitesimal rotation

$$Uf = -y \frac{\partial f}{\partial x} + x \frac{\partial f}{\partial y},$$

a solution of $Uf = 0$ is $x^2 + y^2$. The path curves of the group of rotations about the origin (§ 8) are therefore circles $x^2 + y^2 = c$. The slope at (x, y) is $-x/y = \eta/\xi$.

Consider the infinitesimal transformation

$$Uf = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}$$

which generates the group of similarity transformations (§ 9). Since a solution of $Uf = 0$ is y/x, the path curves are the straight lines through the origin.

13. Corresponding ordinary and partial differential equations. If $v(x, y) = c$ is an integral of the ordinary differential equation (31), we saw that v is a solution of the corresponding linear partial differential equation

$$\xi(x, y) \frac{\partial f}{\partial x} + \eta(x, y) \frac{\partial f}{\partial y} = 0. \tag{34}$$

Conversely, if a function $v(x, y)$, which is not a constant, is a solution of (34), so that (33) holds, then the latter together with the result (32) of differentiating $v = c$ show that dx and dy are proportional to ξ and η, whence $v = c$ is an integral of (31).

THEOREM. *If $v(x, y) = c$ is an integral of the ordinary differential equation* (31), *then v is a solution of the corresponding linear partial differential equation* (34), *and conversely.*

CHAPTER II.

Ordinary differential equations of the first order.

The problem of integrating a differential equation will be reduced to the problem of finding a one-parameter group which leaves the equation unaltered. Such a group presents itself immediately when the equation is linear or homogeneous or of certain other standard types, but especially for differential equations arising in problems in geometry and mechanics (§§ 16, 18—20). In the latter cases we do not attempt to integrate the equation ignoring the data of the problem in which it arose, but rely on the data to suggest a group which leaves the equation unaltered.

14. **First criterion for the invariance of a differential equation under a group.** If the integral curves $\Phi(x, y) = c$ of a differential equation of the first order are merely permuted among themselves (or are individually unaltered) by every transformation of a one-parameter group, the differential equation is said to be *invariant* under the group. The phrase "invariant under the group generated by the infinitesimal transformation Uf" will usually be abbreviated to "invariant under Uf".

LEMMA. *Consider a family of curves $\omega(x, y) = c$ not identical with the family of path curves of the group generated by the infinitesimal transformation Uf. The curves $\omega = c$ are permuted among themselves by every transformation of the group if and only if $U\omega$ is a function of ω.*

First, if the curves $\omega(x, y) = c$ are permuted by every transformation T_t of the group, then to each pair of values of t and c must correspond a value $\gamma(t, c)$ such that

$$\omega(x_1, y_1) = \gamma(t, c) \tag{1}$$

for all sets of solutions x, y of $\omega(x, y) = c$. Then by (13) of § 8,

$$U\omega(x, y) = \left[\frac{\partial \gamma(t, c)}{\partial t}\right]_{t=0},$$

which is a function of c and hence of ω.

Conversely, let $U\omega = F(\omega)$. Since $\omega = c$ is not a path curve, $U\omega$ is not zero identically by Theorem 2 of § 12. By (12) of § 8,

$$\frac{d\omega_1}{dt} = F(\omega_1), \qquad \omega_1 = \omega(x_1, y_1).$$

Since $F(\omega_1)$ is not zero identically, there is a region in which the integral

$$I(\omega_1) = \int^{\cdot} \frac{d\omega_1}{F(\omega_1)}$$

exists. Evidently $I(\omega_1) = t + k$, where k is a constant. This becomes $I(\omega) = k$ for $t = 0$. Hence $I(\omega_1) = t + I(\omega)$, whose solved form is of type (1). This proves the lemma.

The path curves $r(x, y) = c$ are individually unaltered by the group and we have $Ur = 0$ by Theorems 1 and 2 of § 12. This result and the lemma together imply the following conclusion:

THEOREM. *An ordinary differential equation of the first order having the integral $\Phi(x, y) = c$ is invariant under (the group generated by) the infinitesimal transformation Uf if and only if $U\Phi$ is a function of Φ.*

15. **Finding an integrating factor.** Let the equation

$$B(x, y)\,dx - A(x, y)\,dy = 0 \tag{2}$$

be invariant under the infinitesimal transformation

$$Uf = \xi(x, y)\frac{\partial f}{\partial x} + \eta(x, y)\frac{\partial f}{\partial y}, \tag{3}$$

and let the path curves of the group generated by Uf be not identical with the integral curves $\Phi = c$ of the differential equation. Then by § 14, $U\Phi = \psi(\Phi) \not\equiv 0$. We can find as follows a function $F(\Phi)$ such that $UF = 1$:

$$1 = UF(\Phi) \qquad F'(\Phi)\cdot U\Phi = F'(\Phi)\cdot\psi(\Phi), \qquad F(\Phi) = \int^{\cdot} \frac{d\Phi}{\psi(\Phi)}.$$

Since $F(\Phi)$, as well as Φ, is an integral of (2), it is a solution of the corresponding partial differential equation (§ 13)

$$Pf = A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y} = 0. \tag{4}$$

Hence

$$A\frac{\partial F}{\partial x} + B\frac{\partial F}{\partial y} = 0, \qquad UF = \xi\frac{\partial F}{\partial x} + \eta\frac{\partial F}{\partial y} = 1,$$

$$(\xi B - \eta A)\frac{\partial F}{\partial x} = B, \qquad (\xi B - \eta A)\frac{\partial F}{\partial y} = -A.$$

The common multiplier is not zero identically since A and B are not both zero identically by (2). Hence

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy = \frac{Bdx - Ady}{\xi B - \eta A}.$$

Since this fraction is an exact differential dF, it is customary to say that (2) has the *integrating factor*

$$\frac{1}{\xi B - \eta A}. \tag{5}$$

THEOREM. *If the differential equation* $Bdx - Ady = 0$ *is invariant under the infinitesimal transformation* Uf *given by* (3) *and if the path curves of the group generated by* Uf *are not identical with the integral curves of the differential equation, then* (5) *is an integrating factor of the equation.*

The condition on the path curves may be replaced by $\xi B - \eta A \not\equiv 0$. For, then $U\Phi \not\equiv 0$, since otherwise Φ would be a common solution of $U\Phi = 0$, $P\Phi = 0$, and, since the determinant $\xi B - \eta A$ of the coefficients is not zero identically,

$$\frac{\partial \Phi}{\partial x} = 0, \qquad \frac{\partial \Phi}{\partial y} = 0$$

and Φ would reduce to a constant, contrary to hypothesis.

COROLLARY. *If* $Bdx - Ady = 0$ *is invariant under* Uf, *it has the integrating factor* (5) *provided the denominator is not zero identically.*

16. **Solution of simple problems.**

PROBLEM 1. Find an infinitesimal transformation which leaves invariant the general linear differential equation

$$y' + R(x)y = Q(x) \tag{6}$$

of the first order, and integrate it by use of the theorem of § 15.

The corresponding "abridged" equation

$$z' + R(x)z = 0$$

has the integral

$$z(x) = e^{-\int R dx}.$$

Then if $y(x)$ is any integral of (6), $y+az$ is evidently an integral when a is any constant. In other words, (6) is invariant under the group of transformations

$$x_1 = x, \qquad y_1 = y + az.$$

By (26) of § 9, the group is generated by the infinitesimal transformation

$$Uf = z(x)\frac{\partial f}{\partial y}.$$

Hence $-1/z$ is an integrating factor of

$$(6^1) \qquad (Q - Ry)dx - dy = 0.$$

The product obtained is therefore an exact differential, $d\Phi$. Hence

$$\frac{\partial \Phi}{\partial y} = \frac{1}{z} = e^{\int R dx}, \qquad \Phi = ye^{\int R dx} + F(x),$$

$$(-Q + Ry)e^{\int R dx} = \frac{\partial \Phi}{\partial x} = yRe^{\int R dx} + F'(x), \qquad F(x) = -\int Qe^{\int R dx} dx.$$

Solving $\Phi = c$ for y, we get

$$y = e^{-\int R dx}\left\{\int Qe^{\int R dx} dx + c\right\}.$$

PROBLEM 2. Find every curve such that the radius vector to any point P on it makes the same angle θ with the tangent at P that it makes with the x-axis.

Let x and y be the rectangular coördinates of P.

Then

$$\tan\theta = \frac{y}{x}, \quad \tan\iota = \frac{dy}{dx}, \quad \iota = 2\theta,$$

$$\frac{dy}{dx} = \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2xy}{x^2-y^2},$$

$$(7) \qquad 2xydx - (x^2-y^2)dy = 0.$$

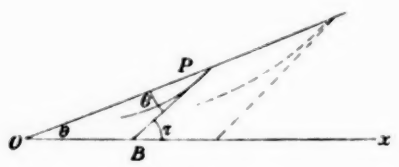

Fig. 3.

If we magnify our figure uniformly from O, we evidently obtain a new curve of the desired family of curves. In other words, each of the transformations

$$(8) \qquad x_1 = ax, \qquad y_1 = ay$$

merely permutes the curves of the family and hence leaves invariant the differential equation (7) of which they are the integral curves. This may be seen also from the fact that (7) is homogeneous. By (26) of § 9, the group of transformations (8) is generated by the infinitesimal transformation

$$Uf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}.$$

Hence an integrating factor of (7) is the reciprocal of

$$x(2xy) - y(x^2 - y^2) = (x^2 + y^2)y.$$

Thus

$$\frac{2x\,dx}{x^2+y^2} - \frac{(x^2-y^2)\,dy}{(x^2+y^2)\,y}$$

is an exact differential, $d\Phi$. Hence

$$\frac{\partial \Phi}{\partial x} = \frac{2x}{x^2+y^2}, \qquad \Phi = \log(x^2+y^2) + F(y),$$

$$\frac{2y}{x^2+y^2} + F'(y) = \frac{\partial \Phi}{\partial y} = \frac{-(x^2-y^2)}{(x^2+y^2)\,y}, \qquad F'(y) = -\frac{1}{y},$$

$$F(y) = -\log y.$$

Thus one integral is $\log(x^2+y^2) - \log y = \text{const.}$, so that another is $x^2 + y^2 = cy$. The required curves are therefore the circles tangent to the x-axis at the origin.

The work becomes simpler if we use polar coördinates. Then

$$\tan OPB = \frac{\varrho\,d\theta}{d\varrho} = \tan\theta, \qquad \tan\theta \cdot d\varrho - \varrho\,d\theta = 0.$$

Since this is invariant under $\varrho_1 = a\varrho$, $\theta_1 = \theta$, and hence under

$$\varrho\frac{\partial f}{\partial \varrho}, \tag{9}$$

an integrating factor is the reciprocal of $\varrho \tan\theta$. The effect is to separate the variables:

$$\frac{d\varrho}{\varrho} - \frac{d\theta}{\tan\theta} = 0, \qquad \log\varrho - \log\sin\theta = C, \qquad \varrho = c\sin\theta.$$

PROBLEM 3. Find every curve such that the radius vector to any point P on it makes a constant angle with the tangent at P.

Denote the tangent of the angle by $1/k$. Then

$$\frac{\rho d\theta}{d\rho} = \frac{1}{k}, \qquad d\rho - k\rho d\theta = 0.$$

Since the curves are evidently merely permuted by the magnification (8), we find from (9) that $1/\rho$ is an integrating factor. The latter follows also by using the infinitesimal rotation $\partial f/\partial\theta$ in the polar coördinates ρ, θ. This generates a group which merely permutes the curves. Multiplication by $1/\rho$ separates the variables and we get the logarithmic spirals $\rho = ce^{k\theta}$.

17. **Infinitude of groups leaving invariant a differential equation.** In the last problem we noticed that a certain differential equation is invariant under two groups. We shall now prove that any differential equation $Bdx - Ady = 0$ is invariant under infinitely many groups. There exists an integrating factor M, whence

$$M(Bdx - Ady) = d\Phi = \frac{\partial\Phi}{\partial x}dx + \frac{\partial\Phi}{\partial y}dy,$$

$$MB = \frac{\partial\Phi}{\partial x}, \qquad MA = -\frac{\partial\Phi}{\partial y}.$$

Choose any pair of functions $\xi(x, y)$, $\eta(x, y)$ such that

$$\frac{1}{B\xi - A\eta} = M, \tag{10}$$

this choice being possible in infinitely many ways. Then

$$1 = MB\xi - MA\eta \qquad \xi\frac{\partial\Phi}{\partial x} + \eta\frac{\partial\Phi}{\partial y} = U\Phi,$$

where Uf is the infinitesimal transformation (3). Since $U\Phi = 1$, the theorem of § 14 shows that our differential equation is invariant under Uf.

THEOREM. *For each of the infinitude of pairs of functions ξ and η satisfying* (10), $Bdx - Ady = 0$ *is invariant under the infinitesimal transformation* $Uf = \xi f_x + \eta f_y$.

18. **Geometrical interpretation of Lie's integrating factor.** Let

$$Bdx - Ady = 0 \tag{11}$$

be invariant under the infinitesimal transformation

$$(12) \qquad Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}.$$

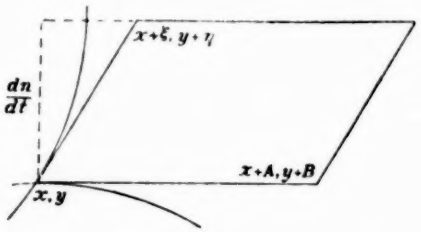

Fig. 4.

Consider the three points (x, y), $(x + A, y + B)$, $(x + \xi, y + \eta)$. The second lies on the tangent to the integral curve through (x, y) of (11), since its slope is B/A. The third point lies on the tangent to the path curve through (x, y) of the group generated by Uf, since its slope is η/ξ by Theorem 2 of § 12. The area of the parallelogram of which these three points are consecutive corners is the absolute value of the determinant

$$\begin{vmatrix} x & y & 1 \\ x+\xi & y+\eta & 1 \\ x+A & y+B & 1 \end{vmatrix} = \begin{vmatrix} x & y & 1 \\ \xi & \eta & 0 \\ A & B & 0 \end{vmatrix} = \xi B - \eta A,$$

whose reciprocal is the integrating factor (5) of § 15.

The area of the parallelogram is also the product of the length $\sqrt{A^2 + B^2}$ of one side by the projection on the normal to the initial integral curve of the length $\sqrt{\xi^2 + \eta^2}$ of the adjacent side. By the differential equations

$$dx = \xi(x, y)\, dt, \qquad dy = \eta(x, y)\, dt$$

of the group, the element of arc along the path curve through (x, y) is

$$\sqrt{dx^2 + dy^2} = \sqrt{\xi^2 + \eta^2} \cdot dt.$$

Let dn denote the projection of this element of arc on the same normal. Then the projection of the side of length $\sqrt{\xi^2 + \eta^2}$ is dn/dt, so that the area of the parallelogram is $\sqrt{A^2 + B^2} \cdot dn/dt$.

THEOREM. *If the differential equation* (11) *is invariant under the infinitesimal transformation* (12), *it has the integrating factor*

$$(13) \qquad \frac{1}{\dfrac{dn}{dt}\sqrt{A^2 + B^2}}.$$

19. **Parallel curves.** If on each normal to a given curve C_0 a segment of constant length t is laid off, their end points form a new curve C_t. When t varies, we obtain a family of curves called *parallel curves*.

An arbitrarily chosen point $P:(x, y)$ determines uniquely a curve C_k of the family. Let N be the normal to C_0 which passes through P. The intersection of N with C_{k+t} is a point P_1 whose coördinates x_1, y_1 are functions of x, y and t. These functions define a transformation T_t which replaces P by P_1. Similarly, the transformation T_t replaces P_1 by the point P_2 of intersection of N with C_{k+t+t}. Since the product $T_t T_t$ replaces P by P_2, we have $T_t T_t = T_{t+t}$. Hence all the transformations T_t form a group whose path curves are the normals to C_0. The element of arc dn on any normal is dt. Hence we may suppress dn/dt from the integrating factor (13).

THEOREM. *A differential equation $Bdx - Ady = 0$ whose integral curves are parallel curves has the integrating factor $1/\sqrt{A^2+B^2}$.*

By the *involutes* of a curve C are meant the curves which are orthogonal to the tangents of C. It is proved in the calculus that the various involutes of C form a system of parallel curves. Hence, the preceding theorem gives an integrating factor of the differential equation of the involutes. This equation is readily obtained.

When C is the circle $x^2+y^2=1$, the equation of the tangent having the slope p is

$$y = px + \sqrt{1+p^2}.$$

Solving for $-1/p$, we get

$$-\frac{1}{p} = \frac{xy \mp \sqrt{x^2+y^2-1}}{1-y^2}.$$

Hence this gives the slope of a tangent to an involute of C. Choosing, for example, the upper sign and equating the fraction to dy/dx, we get the following differential equation of the involutes:

$$(xy+\sqrt{x^2+y^2-1})\,dx + (y^2-1)\,dy = 0.$$

Here A^2+B^2 is seen to be a perfect square and an integrating factor is the reciprocal of $x+y\sqrt{x^2+y^2-1}$. The usual method of integration yields the equation of the involutes:

$$\sqrt{x^2+y^2-1} + \sin^{-1}\frac{1}{\sqrt{x^2+y^2}} - \tan^{-1}\frac{y}{x} = c.$$

20. **Isothermal curves.** Let $\omega(x, y) = c$ be the integral curves of

$$Bdx - Ady = 0. \tag{14}$$

They are cut orthogonally by the integral curves $\lambda(x, y) = c'$ of

$$Adx + Bdy = 0. \tag{15}$$

Through an arbitrarily chosen point P: (x, y) passes a unique integral curve $\omega(x, y) = k$ of (14) and a unique one N of (15). The intersection of N with $\omega(x, y) = k + t$ is a point P_1 whose coördinates x_1, y_1 are functions of x, y, t. These functions define a transformation T_t which replaces P by P_1. Similarly $T_{t'}$ replaces P_1 by the point P_2 of intersection of N with $\omega(x, y) = k + t + t'$. Then $T_t T_{t'}$ and $T_{t+t'}$ are identical since each replaces P by P_2. Hence all the T_t form a group whose path curves are the integral curves of (15).

Interchanging the rôles of ω and λ, we get a second group of transformations T_t' whose path curves are the integral curves of (14).

If the ratio ds/dt of the element of arc ds of a path curve to dt is the same function of x and y in both cases, we call either family of curves a system of *isothermal** curves. Since also the sum of the squares of the coefficients in either (14) or (15) is $A^2 + B^2$, it follows from the theorem in § 18 that the two differential equations have a common integrating factor M. We proceed to show that M can be found by quadratures (the integration of a function of a single variable being called a quadrature).

Since the product of (15) by M is an exact differential, say dq,

$$M(Adx + Bdy) \qquad \frac{\partial q}{\partial x}dx + \frac{\partial q}{\partial y}dy, \qquad MA = \frac{\partial q}{\partial x}, \qquad MB = \frac{\partial q}{\partial y},$$

$$\frac{\partial (MA)}{\partial y} = \frac{\partial (MB)}{\partial x}, \tag{16}$$

each member being a value of $\partial^2 q / \partial x \, \partial y$. Hence (16) is the condition that $MAdx + MBdy$ be an exact differential.

Performing the differentiations in (16) and dividing by M, we get

$$A\frac{\partial \log M}{\partial y} - B\frac{\partial \log M}{\partial x} = \frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}.$$

To derive the corresponding result for (14), we have only to replace A by B and B by $-A$; hence

$$B\frac{\partial \log M}{\partial y} + A\frac{\partial \log M}{\partial x} = -\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y}.$$

* Thus a family of isothermal curves, together with the curves cutting them orthogonally, divide the plane into "infinitesimal squares". For example, all concentric circles form a family of isothermal curves; they are cut orthogonally by the radii.

Solving the last two conditions as linear equations, we get

$$\frac{\partial \log M}{\partial x} = \frac{B\frac{\partial A}{\partial y} - A\frac{\partial B}{\partial y} - A\frac{\partial A}{\partial x} - B\frac{\partial B}{\partial x}}{A^2+B^2}$$

or

$$\frac{\partial \log M}{\partial x} = -\frac{\partial}{\partial y}\tan^{-1}\frac{B}{A} - \tfrac{1}{2}\frac{\partial}{\partial x}\log(A^2+B^2), \tag{17}$$

$$\frac{\partial \log M}{\partial y} = \frac{A\frac{\partial B}{\partial x} - B\frac{\partial A}{\partial x} - A\frac{\partial A}{\partial y} - B\frac{\partial B}{\partial y}}{A^2+B^2}$$

or

$$\frac{\partial \log M}{\partial y} = \frac{\partial}{\partial x}\tan^{-1}\frac{B}{A} - \tfrac{1}{2}\frac{\partial}{\partial y}\log(A^2+B^2). \tag{18}$$

Multiplying (17) by dx and (18) by dy and adding, we get the value of $d\log M$, so that $\log M$ can be found by quadratures and then M is obtained. The condition that the value of $d\log M$ be an exact differential is (as shown above) that the partial derivative of (17) with respect to y be identical with that of (18) with respect to x, and hence is

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = 0, \qquad t = \tan^{-1}\frac{B}{A}. \tag{19}$$

Conversely, when condition (19) is satisfied, equations (14) and (15) have a common integrating factor and (13) shows that du/dt is the same at any point for the two families. Hence the integral curves of (14) are isothermal curves.

THEOREM. *The integral curves of* $Bdx - Ady = 0$ *are isothermal if and only if condition* (19) *is satisfied.*

For the concentric circles with center at the origin, whose differential equation is evidently $xdx + ydy = 0$, we have $B/A = -x/y$, and (19) is seen to be satisfied. From (17) and (18), we readily get $M = 1/(x^2+y^2)$, which is a common integrating factor of the differential equation of the circles and that of the radii, $ydx - xdy = 0$.

Again, the family of circles tangent to the x-axis at the origin, $x^2+y^2-cy=0$, has the differential equation $2xydx - (x^2-y^2)dy = 0$. Thus

$$t = \tan^{-1}\frac{B}{A} = \tan^{-1}\frac{2xy}{x^2-y^2} = 2\tan^{-1}\frac{y}{x}.$$

Hence by the preceding example, condition (19) is here satisfied. The differential equation of the family of curves orthogonal to the above circles is $(x^2-y^2)dx + 2xydy = 0$, whose

integral curves are the circles $x^2+y^2-cx=0$ tangent to the y-axis at the origin. This is evident since the second differential equation can be derived from the first by interchanging x and y: or we may employ the common integrating factor

$$M = \frac{1}{(x^2+y^2)^2},$$

which is easily derived from (17) and (18).

Readers interested in the differential geometry of surfaces will find applications of the above and similar principles to minimal, isothermal, or asymptotic lines, and lines of curvature of surfaces, in Lie-Scheffers, Differentialgleichungen, pp. 160–187 (or the resumé in Cohen's Lie Theory of One-Parameter Groups, pp. 78–82).

21. **Commutator.** Given two linear partial differential expressions

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}, \tag{20}$$

$$Pf = A \frac{\partial f}{\partial x} + B \frac{\partial f}{\partial y}, \tag{21}$$

we define the *commutator* (alternant or Klammerausdruck) of Uf with Pf to be

$$(UP)f = U(Pf) - P(Uf),$$

and shall prove that, after cancellations of second derivatives, it reduces to a third linear partial differential expression. We have

$$\left.\begin{array}{l} U(Pf) \\ -P(Uf) \end{array}\right\} \quad \left\{\begin{array}{l} \xi \dfrac{\partial}{\partial x}\left(A \dfrac{\partial f}{\partial x} + B \dfrac{\partial f}{\partial y}\right) + \eta \dfrac{\partial}{\partial y}\left(A \dfrac{\partial f}{\partial x} + B \dfrac{\partial f}{\partial y}\right) \\ -A \dfrac{\partial}{\partial x}\left(\xi \dfrac{\partial f}{\partial x} + \eta \dfrac{\partial f}{\partial y}\right) - B \dfrac{\partial}{\partial y}\left(\xi \dfrac{\partial f}{\partial x} + \eta \dfrac{\partial f}{\partial y}\right). \end{array}\right.$$

If each time we differentiate the second factor of the products, we get the terms

$$\begin{array}{l} \xi A \dfrac{\partial^2 f}{\partial x^2} + \xi B \dfrac{\partial^2 f}{\partial x \partial y} + \eta A \dfrac{\partial^2 f}{\partial y \partial x} + \eta B \dfrac{\partial^2 f}{\partial y^2} \\ -A\xi \dfrac{\partial^2 f}{\partial x^2} - A\eta \dfrac{\partial^2 f}{\partial x \partial y} - B\xi \dfrac{\partial^2 f}{\partial y \partial x} - B\eta \dfrac{\partial^2 f}{\partial y^2}, \end{array}$$

which cancel in pairs. Next, if we differentiate the first factor of the products, we get

$$\left(\xi\frac{\partial A}{\partial x}+\eta\frac{\partial A}{\partial y}-A\frac{\partial\xi}{\partial x}-B\frac{\partial\xi}{\partial y}\right)\frac{\partial f}{\partial x}$$
$$+\left(\xi\frac{\partial B}{\partial x}+\eta\frac{\partial B}{\partial y}-A\frac{\partial\eta}{\partial x}-B\frac{\partial\eta}{\partial y}\right)\frac{\partial f}{\partial y},$$

which may be written in the form

$$(UP)f=(UA-P\xi)\frac{\partial f}{\partial x}+(UB-P\eta)\frac{\partial f}{\partial y}. \tag{22}$$

Hence, the commutator $(UP)f$ is found by the following simple rule. The first coefficient (that of $\partial f/\partial x$) of the commutator is found by applying the operator Uf to the first coefficient A of Pf and subtracting the result of applying the operator Pf to the first coefficient of Uf, and similarly for the second coefficient.

22. **A second criterion for the invariance of a differential equation under an infinitesimal transformation.** Our former criterion (§ 14) depended upon a knowledge of an integral of the equation. Since an integral is not known in advance, but is just what we are seeking, that criterion was mainly of theoretical value. We shall now establish another criterion which does not assume the knowledge of an integral and can be tested at once on any given equation.

Let Φ be an integral of the differential equation and let the corresponding partial differential equation be

$$Pf=A\frac{\partial f}{\partial x}+B\frac{\partial f}{\partial y}=0, \tag{23}$$

so that $P\Phi=0$.

Assume that the equation is invariant under Uf defined by (20). Then $U\Phi$ is a function $\psi(\Phi)$ of Φ by § 14. Hence

$$(UP)\Phi=U(P\Phi)-P(U\Phi)=U(0)-P(\psi(\Phi))=0,$$

so that Φ is a solution of the linear partial differential equation obtained by equating (22) to zero, as well as of (23). Hence the coefficients of the former are proportional to those of the latter and $(UP)f=\sigma Pf$, where σ is some function of x and y.

Conversely, $(UP)f=\sigma Pf$ implies

$$0=(UP)\Phi=U(P\Phi)-P(U\Phi),\qquad 0=P(U\Phi),$$

whence $U\Phi$, being a solution of $Pf = 0$, is a function of Φ. Hence (§ 14), our initial differential equation is invariant under Uf.

THEOREM.* *An ordinary differential equation is invariant under the infinitesimal transformation Uf if and only if $(UP)f = \sigma Pf$, where $Pf = 0$ is the corresponding partial differential equation.*

For example, consider the linear differential equation

$$y' + R(x)y = Q(x), \quad \frac{dx}{1} = \frac{dy}{Q - Ry}.$$

The corresponding partial differential equation is

$$Pf = \frac{\partial f}{\partial x} + (Q - Ry)\frac{\partial f}{\partial y} = 0.$$

We seek a function $z(x)$ such that $Pf = 0$ is invariant under the infinitesimal transformation of the special type

$$Uf = z(x)\frac{\partial f}{\partial y}$$

employed in § 16. Since

$$(UP)f = (-zR - z')\frac{\partial f}{\partial y}$$

lacks $\partial f / \partial x$, it will be a multiple of Pf only if it is zero identically, whence z is an integral of the abridged equation (§ 16).

23. **Group of extended transformations.** The point transformation

$$T_a: \qquad x_1 = \Phi(x, y, a), \qquad y_1 = \psi(x, y, a) \tag{24}$$

transforms any direction $y' = dy/dx$ into the new direction

$$\frac{dy_1}{dx_1} = y_1' = \frac{\psi_x + \psi_y y'}{\Phi_x + \Phi_y y'}. \tag{25}$$

The combination of a point (x, y) and a straight line (of slope y') through it is called a *lineal element*. The three equations (24) and (25) define a transformation T_a'' of lineal elements which is called the (first) *extended transformation* on the three variables x, y, y'.

These extended transformations form a group if the transformations (24) form a group. For, if T_a, T_b and T_c are identical transformations (24) and hence take any curve C_1 into the same curve C_2, the extended transformations T_a'', T_b'' and T_c'' both take the points and slopes of the curve C_1 into the points and slopes of C_2 and hence are identical transformations.

* Another proof of this theorem is given at the end of § 28.

Thus the product of any two extended transformations T'_a and T'_b is an extended transformation T'_c. Next, if (24) is the identity transformation when $a = a_0$, then $x = \Phi(x, y, a_0)$, $y = \psi(x, y, a_0)$, whence

$$\Phi_x = 1, \quad \Phi_y = 0, \quad \psi_x = 0, \quad \psi_y = 1, \text{ when } a = a_0. \tag{26}$$

Thus (25) reduces to $y'_1 = y'$ for $a = a_0$, so that T'_{a_0} is the identity transformation on x, y, y'. Finally, if T_a takes any curve C into the curve C_1, the inverse transformation T_{a_1} takes C_1 into C, whence $T'_a T'_{a_1}$ leaves invariant each point and each slope of C and is therefore the identity transformation, so that T'_a has the inverse T'_{a_1}.

By (26) of § 9, the infinitesimal transformation of the group (24) is

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}, \quad \xi = \left[\frac{d\Phi(x, y, a)}{da} \right]_{a = a_0}, \quad \eta = \left[\frac{d\psi}{da} \right]_{a = a_0}.$$

To find the infinitesimal transformation of the extended group, we differentiate equations (24) and (25) with respect to a and then take $a = a_0$. This was done above for (24). Next, denote the fraction (25) by N/D. Then, by (26),

$$N = y', \quad D = 1, \quad \frac{dN}{da} = \eta_x + \eta_y y' = \frac{d\eta}{dx}, \quad \frac{dD}{da} = \xi_x + \xi_y y' = \frac{d\xi}{dx},$$

when $a = a_0$. Write η' for the value of the derivative of N/D when $a = a_0$. Then

$$\eta' = \frac{d\eta}{dx} - y' \frac{d\xi}{dx} = \eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2. \tag{27}$$

The symbol of the infinitesimal transformation of the extended group is therefore

$$U'f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}. \tag{28}$$

24. **Invariants.** A function $f(x, y)$ is said to be invariant under a transformation on x and y if it remains unchanged by the transformation. Thus $f(x, y)$ is invariant under a group of transformations with the parameter a if and only if $f(x_1, y_1) = f(x, y)$ for all values of x, y, a. Then $df(x_1, y_1)/da = 0$. Conversely, this implies that $f(x_1, y_1)$ is independent of a and hence equal to its value $f(x, y)$ for $a = a_0$, whence $x_1 = x$, $y_1 = y$. Hence by § 8,

$f(x, y)$ is invariant under the group generated by the infinitesimal transformation Uf if and only if Uf is identically zero in x and y.

For example, x^2+y^2 is invariant under every rotation about the origin O since the distance $\sqrt{x^2+y^2}$ of the point (x, y) from O is unchanged by the rotation. The infinitesimal rotation is

$$Uf = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y},$$

and x^2+y^2 is invariant under the group of rotations since

$$U(x^2+y^2) = -y(2x) + x(2y) \equiv 0.$$

25. **Differential invariants.** A function $F(x, y, y')$ which actually involves the derivative y' is called a *differential invariant* of the first order of a group of point transformations (24) if, when F is regarded as a function of three independent variables, it is invariant under the group of the first extended transformations defined by (24) and (25). This will be the case if and only if $U'F$ is identically zero in x, y, y', where $U'f$ is the infinitesimal transformation (28) of the extended group. In fact, the discussion of invariants given in § 24 holds true for three or more independent variables as well as for two.

For example, let Uf be the infinitesimal transformation

$$Uf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} \tag{29}$$

of the group of magnifications from the origin O (similarity transformations). By (27), $\eta' = 0$, whence $U'f = Uf$. Hence any function of y' is invariant under $U'f$ and is therefore a differential invariant of Uf. This is obvious geometrically since the slope y' of any straight line through the origin O is unaltered by a magnification from O.

26. **Determination of all differential invariants of the first order of Uf.** We desire the solutions, involving y', of

$$U'f = \xi\frac{\partial f}{\partial x} + \eta\frac{\partial f}{\partial y} + \eta'\frac{\partial f}{\partial y'} = 0. \tag{30}$$

The corresponding system of ordinary differential equations is

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dy'}{\eta'}. \tag{31}$$

The equation $dx/\xi = dy/\eta$, in x and y only, has an integral $u(x, y) = c$. Solve the latter for y in terms of x and c, and substitute the value of y in $dx/\xi = dy'/\eta'$. Employing the expression (27) for η', we get

$$\frac{dy'}{dx} = p + qy' + ry'^2,$$

where p, q, r are functions of x and c. This Riccati differential equation of the first order has an integral $\Phi(x, y', c) = k$, where k is an arbitrary constant. Replacing c by $u(x, y)$, we obtain a second integral $v(x, y, y') = k$ of the system (31).

By differentiation of $v = k$ we have

$$\frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial y'}dy' = 0.$$

Replacing dx, dy, dy' by the expressions ξ, η, η' which are proportional to them by (31), we get

$$\xi\frac{\partial v}{\partial x} + \eta\frac{\partial v}{\partial y} + \eta'\frac{\partial v}{\partial y'} = 0,$$

which shows that v is a solution of (30). Similarly, u is a solution. Also an arbitrary function $\psi(u, v)$ of u and v is a solution since

$$U'\psi = \frac{\partial \psi}{\partial u}U'u + \frac{\partial \psi}{\partial v}U'v = 0.$$

Furthermore, every solution of (30) is of the form $\psi(u, v)$. In fact, the solutions u and v are independent, since v involves y' while u is free of y'. If there were a solution w independent of u and v, then x, y, y' could be expressed as functions of u, v, w and hence would be solutions of (30), whence ξ, η, η' would be identically zero, contrary to hypothesis.*

THEOREM. *All differential invariants of the first order of the group generated by Uf are functions involving y' of two independent solutions $u(x, y)$ and $v(x, y, y')$ of $U'f = 0$ which may be found by integrating two ordinary differential equations of the first order in two variables.*

* To give another proof, introduce u, v, w as new variables. By § 29, the coefficients of the new partial differential equation are identically zero. Returning to the initial variables, we conclude that the coefficients of (30) are all identically zero.

For example, when Uf is given by (29), we may evidently take $u = y/x$, $v = y'$. Hence every differential invariant is a function of y/x and y'.

Again, to find all differential invariants of

$$Uf = ny\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}, \tag{32}$$

which is the infinitesimal rotation about the origin if $n = -1$, we note that $\eta' = 1 - ny'^2$ by (27), so that

$$U'f = ny\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} + (1 - ny'^2)\frac{\partial f}{\partial y'}. \tag{33}$$

The corresponding system of ordinary differential equations is

$$\frac{dx}{ny} = \frac{dy}{x} = \frac{dy'}{1 - ny'^2}.$$

An evident integral is $u = x^2 - ny^2$. To find a second integral v, denote each of the three equal fractions by f. Then

$$dx = nyf, \qquad dy = xf, \qquad \frac{x\,dy - y\,dx}{x^2 - ny^2} = f = \frac{dy'}{1 - ny'^2}.$$

Write $n = -c^2$. Then an integral is $\tan^{-1} cy/x - \tan^{-1} cy'$, whose tangent is the product of c by

$$v = \frac{y - xy'}{x - nyy'}.$$

27. **Invariant equations.** We shall call $\omega(x, y) = 0$ invariant under the group G generated by Uf if $\omega(x_1, y_1) = 0$ for all values of t and all values of x, y which satisfy $\omega(x, y) = 0$. Then the derivative of $\omega(x_1, y_1)$ with respect to t must be zero for those values. Hence by (13) of § 8, $U'\omega(x, y)$ must be zero for all values of x, y satisfying $\omega(x, y) = 0$.

Conversely, let this condition hold. In case $\xi = \eta = 0$ for every point (x, y) on the curve $\omega = 0$, each point and hence the curve is invariant under G (§ 11). We shall exclude the singular case in which the derivatives ω_x and ω_y are both zero at every point on $\omega = 0$. Let (x, y) be any point on the curve for which those derivatives are not both zero, and also $\xi \neq 0$. Since

$$U'\omega = \omega_x \xi + \omega_y \eta = 0$$

at (x, y), we have $\omega_y \neq 0$. Hence for the slope dy/dx of the curve $\omega = 0$ at (x, y), we have

$$\omega_x dx + \omega_y dy = 0, \qquad \frac{dy}{dx} = -\frac{\omega_x}{\omega_y} = \frac{\eta}{\xi}.$$

But η/ξ is the slope of a path curve of the group by Theorem 2 of § 12. Since the differential equation $dy/dx = \eta/\xi$ has a unique integral curve

passing through (x, y), and $\omega = 0$ is one solution, it follows that $\omega = 0$ is a path curve or an aggregate of path curves and curves each of whose points is invariant under G. Since each path curve is invariant (§ 12), $\omega = 0$ is an invariant curve.

THEOREM. *A curve $\omega(x, y) = 0$ is invariant under the group generated by Uf if and only if $U\omega$ is zero for all values of x and y which satisfy $\omega = 0$. A like result holds for surfaces $\omega(x, y, z) = 0$ under transformations on x, y, z.*

To prove the last part of the theorem, suppose that

$$U\omega(x, y, z) \equiv \xi\omega_x + \eta\omega_y + \zeta\omega_z = 0 \tag{34}$$

for all values of x, y, z which satisfy $\omega = 0$. Let (x, y, z) be any point on the surface $\omega = 0$ for which ξ, η, ζ are not all zero, say $\xi \neq 0$. Since $\omega_x, \omega_y, \omega_z$ are proportional to the direction cosines of the normal to the surface $\omega = 0$ at (x, y, z), it follows from (34) that ξ, η, ζ are proportional to the direction cosines of a tangent to the surface at (x, y, z). But the direction cosines of the tangent to a path curve of the group generated by Uf are proportional to dx, dy, dz and hence to ξ, η, ζ. Thus at each non-invariant point of the surface $\omega = 0$ the tangent to the path curve is a tangent to the surface. It may be shown as follows that the path curve lies on the surface. We may take $\omega = 0$ in the solved form $z = z(x, y)$. From the differential equation

$$\frac{dy}{dx} = \frac{\eta(x, y, z(x, y))}{\xi(x, y, z(x, y))},$$

we get $y = y(x, c)$, where c is an arbitrary constant. From this and $z = z(x, y(x, c))$, we get

$$\frac{dy}{dx} = \frac{\eta}{\xi}, \qquad \frac{dz}{dx} = z_x + z_y\frac{\eta}{\xi} = \frac{\zeta}{\xi},$$

since (34) becomes $\xi z_x + \eta z_y - \zeta = 0$ for $\omega = z(x, y) - z$. Hence the unique (path) curve through (x, y, z) for which $dx : dy : dz = \xi : \eta : \zeta$ is on the surface $\omega = 0$. Since $\omega = 0$ is the locus of an infinitude of path curves, it is invariant under the group.

28. **A third criterion for the invariance of a differential equation under Uf.** Under a point transformation (24) and the induced transformation (25) on y', a differential equation $F(x, y, y') = 0$ is transformed into an equation $F_1(x_1, y_1, y_1') = 0$, and the integral curves $\Phi(x, y) = c$ of the former go into certain curves $\Phi_1(x_1, y_1) = c$. Since any set of

solutions x, y, y' of $F = 0$ goes into a set of solutions x_1, y_1, y_1' of $F_1 = 0$, while any point (x, y) and slope y' of the curve $\Phi = c$ go into a point (x_1, y_1) and slope y_1' of $\Phi_1 = c$, it is evident that the transformed curves $\Phi_1 = c$ are integral curves of $F_1 = 0$. Hence *the integral curves of a differential equation are transformed by any point transformation into the integral curves of the transformed equation.*

In particular, the differential equation $F = 0$ is invariant under a group of point transformations if and only if the ordinary equation $F = 0$ in three independent variables x, y, y' is invariant under the group of extended transformations defined by (24) and (25). The condition for the latter is given by the theorem in § 27.

THEOREM. *A differential equation $F(x, y, y') = 0$ is invariant under the group of point transformations generated by an infinitesimal transformation Uf if and only if $U'f$ vanishes for all sets of values x, y, y' satisfying $F = 0$, where $U'f$ is the first extension* (28) *of Uf.*

For example, let Uf be the infinitesimal rotation given by (32) for $n = -1$, so that $U'f$ is (33). The lines $yx = c$ are merely permuted by a rotation about the origin. They are the integral curves of the differential equation $xy' - y = 0$, which is therefore invariant under Uf. This fact also follows from the present theorem since

$$U'(xy' - y) \equiv -yy' - x + (1 + y'^2)x \equiv y'(xy' - y).$$

Again, the differential equation of the tangents to $x^2 + y^2 = 1$ is

$$F \equiv 1 + y'^2 - (y - xy')^2 = 0,$$

which is invariant under the same Uf since $U'F \equiv 2y'F$.

Let us apply the preceding theorem to the case in which F has the solved form $F \equiv Ay' - B = 0$, where A and B are functions of x and y alone. We have

$$U'f \equiv Uf + \eta' \frac{\partial f}{\partial y'}, \qquad U'F \equiv y' UA - UB + \eta' A.$$

We are to employ all sets of values $x, y, y' = B/A$ for which $F = 0$. For $y' = B/A$, the expression (27) for η' becomes

$$\frac{1}{A}(A\eta_x + B\eta_y) - \frac{B}{A^2}(A\xi_x + B\xi_y) \equiv \frac{1}{A} P\eta - \frac{B}{A^2} P\xi,$$

where $Pf \equiv Af_x + Bf_y$; and the quotient of $U'F$ by B becomes

$$\frac{1}{A}(UA - P\xi) - \frac{1}{B}(UB - P\eta),$$

which, by the theorem, shall be zero identically in x and y. Then by (22), the coefficients of $(UP)f$ are proportional to those of Pf. This gives a new proof of the theorem of § 22 which states that $Ady - Bdx = 0$ is invariant under Uf if and only if $(UP)f \equiv \sigma Pf$, where σ is a function of x and y alone.

29. **Introduction of new variables in a linear partial differential expression.** We shall express

$$Pf = A_1 \frac{\partial f}{\partial x_1} + \cdots + A_n \frac{\partial f}{\partial x_n}$$

in terms of new variables $y_1, \ldots, y_n$ which are independent functions of $x_1, \ldots, x_n$. Since we may express $f(x_1, \ldots, x_n)$ as a function of $y_1, \ldots, y_n$, we have

$$\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial y_1}\frac{\partial y_1}{\partial x_i} + \cdots + \frac{\partial f}{\partial y_n}\frac{\partial y_n}{\partial x_i}.$$

Multiplying by A_i and summing for $i = 1, \ldots, n$, we get

$$Pf \equiv \frac{\partial f}{\partial y_1}\sum_{i=1}^{n} A_i \frac{\partial y_1}{\partial x_i} + \cdots + \frac{\partial f}{\partial y_n}\sum_{i=1}^{n} A_i \frac{\partial y_n}{\partial x_i}.$$

These sums are equal to $Py_1, \ldots, Py_n$ respectively.

THEOREM. *If in any linear partial differential expression Pf in n variables we introduce n new independent variables $y_1, \ldots, y_n$, we obtain*

$$[Py_1]\frac{\partial f}{\partial y_1} + \cdots + [Py_n]\frac{\partial f}{\partial y_n},$$

where $[Py_i]$ denotes the function obtained by expressing Py_i in terms of $y_1, \ldots, y_n$.

For example, let Pf and the new variables ρ, θ be

$$Pf = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}, \qquad \rho = \sqrt{x^2+y^2}, \qquad \theta = \tan^{-1}\frac{y}{x}.$$

Then $P\rho = 0$, $P\theta = 1$, so that the infinitesimal rotation Pf about the origin becomes $\partial f/\partial\theta$ when expressed in polar coördinates.

30. **Determination of all differential equations invariant under a given infinitesimal transformation Uf.** By § 28, $F(x, y, y') = 0$ is invariant under Uf if and only if $U'F$ vanishes for all sets of values x, y, y' satisfying $F = 0$. In particular, $F = 0$ is invariant if $U'F$ is

identically zero in x, y, y'; then F is a differential invariant of Uf (§ 25) and is a function of two solutions $u(x, y)$ and $v(x, y, y')$ of $U'f = 0$ (§ 26). Conversely, if we equate to zero any function, actually involving v, of u and v, we obtain a differential equation invariant under Uf, since u, v and any function of them are invariant.

Are there differential equations $F = 0$ invariant under Uf which are not expressible in the form $\psi(u, v) = 0$? It seems plausible that $U'F$ may vanish for all sets of solutions of $F = 0$ without being identically zero. For if

$$Vf = x\frac{\partial f}{\partial x}, \qquad F = x$$

$VF = x$ is not identically zero and yet VF vanishes when $F = 0$.

In answering our question we may assume without loss of generality that $u(x, y)$ is not free of y (otherwise, we repeat the following discussion with x and y interchanged and ξ and η interchanged). This assumption and the fact that u is a solution of $Uf = 0$ imply that ξ is not identically zero. Then x, u, v are independent functions of x, y, y' and may be introduced as new variables; by § 29, $U'f$ becomes

$$U'x \cdot \frac{\partial f}{\partial x} + U'u \cdot \frac{\partial f}{\partial u} + U'v \cdot \frac{\partial f}{\partial v} = \lambda\frac{\partial f}{\partial x},$$

where λ denotes the function obtained by expressing $U'x = \xi$ in terms of the new variables. Let $F = 0$ become $\sigma(x, u, v) = 0$ when expressed in terms of the new variables. If x does not actually occur in σ, we have an equation in u and v mentioned above. Suppose next that x occurs in $\sigma = 0$, which may therefore be written in the solved form $x - \Phi(u, v) = 0$. Since it is invariant under $\lambda\,\partial f/\partial x$ by hypothesis, λ is zero for all sets of solutions of $x - \Phi = 0$. Returning to the initial variables, we conclude that ξ is zero for all sets of solutions of $F = 0$. But the differential equation $F = 0$ involves y' and serves to determine y' as a function $f(x, y)$ of x and y. Hence ξ is zero for all sets of values $x, y, y' = f(x, y)$. Since y' does not occur in ξ, ξ is zero for all values of x, y, contrary to the fact that ξ is not zero identically in x, y.

THEOREM. *A differential equation of the first order is invariant under the group generated by the infinitesimal transformation Uf if and only if it can be expressed as an ordinary equation between two independent solutions u and v of $U'f = 0$.*

Two examples were discussed at the end of § 26. The first appears as the case $s = 1$, $r = -1$ of line 2 of the following table, while the

second appears in line 6. The result (§ 16) for a linear differential equation appears as the case $s = 0$ of line 3. The further entries in the table were obtained by starting with infinitesimal transformations so chosen that we can perform the integrations necessary to compute the invariants u and v.

TABLE OF DIFFERENTIAL EQUATIONS INVARIANT UNDER THE ACCOMPANYING INFINITESIMAL TRANSFORMATIONS.

Notations: $p = \frac{\partial f}{\partial x}$, $q = \frac{\partial f}{\partial y}$, n, r, s constants.

$$y' = f(rx + sy), \qquad sp - rq.$$

$$xy' = yf(x^r y^s), \qquad sxp - ryq.$$

$$y' + R(x)y = Q(x)y^s, \qquad e^{\int (s-1)R\,dx} y^s q.$$

$$xy' - ny = g(x) f\left(\frac{y}{x^n}\right), \qquad \frac{x^n}{g(x)}(xp + nyq).$$

$$\frac{x^r y^s (xy' - ny)}{s\,xy' - ry} = f\left(\frac{y}{x^n}\right), \qquad x^r y^s (xp + nyq).$$

$$v \equiv \frac{y - xy'}{x - nyy'} = f(x^2 - ny^2) \;\therefore\; \sqrt{\frac{x^2 - ny^2}{1 - nv^2}} \equiv \frac{x - nyy'}{\sqrt{1 - ny'^2}}$$

$$= g(x^2 - ny^2), \quad nyp + xq.$$

$$\therefore\; 1 \pm \sqrt{n}\,v \equiv \frac{(x \pm \sqrt{n}y)(1 \mp \sqrt{n}y')}{x - nyy'} = h(x^2 - ny^2),$$

$$\frac{x + \sqrt{n}y}{x - \sqrt{n}y} \cdot \frac{1 - \sqrt{n}y'}{1 + \sqrt{n}y'} = k(x^2 - ny^2).$$

$$yy' \pm x = g(x) f(x^2 \pm y^2), \qquad \frac{1}{g(x)}\left(p \mp \frac{x}{y} q\right).$$

$$x^{1-r} y^{s-1} y' = f\left(\frac{x^r}{r} \pm \frac{y^s}{s}\right), \qquad x^{1-r} p \mp y^{1-s} q.$$

Further types may be derived from these by interchanging x with y and hence replacing y' by $1/y'$. All further types listed in the books by Cohen and Page are special cases of the foregoing. This table serves as a key to the integration of differential equations of the first order.

CHAPTER III.

Solution of one or more linear partial differential equations of the first order.

After presenting standard methods of solving a single linear partial differential equation in three variables, we shall give the simpler methods available when the equation is invariant under one or two known infinitesimal transformations. The latter methods are based on the classic theory of complete systems, which is developed in full detail for the typical case of two equations in three variables (the only case required in this paper except in the discussion of covariants in § 55). In particular, this case is sufficient for the important applications in chapter IV to ordinary differential equations of the second order.

31. **Existence of solutions.** To the equation

$$(1) \qquad Pf = A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y} + C\frac{\partial f}{\partial z} = 0$$

in which A, B, C are functions of x, y, z, we make correspond the system of ordinary differential equations

$$(2) \qquad \frac{dx}{A} = \frac{dy}{B} = \frac{dz}{C}.$$

If $v(x, y, z) = \text{const.}$ is any integral of (2), it was readily verified in § 26 that v is a solution of (1). Also, if u and v are two independent solutions, all functions of u and v are solutions and exhaust the solutions of (1). The relation between the equations (1) and (2) in three variables is analogous to that established in § 13 for equations in two variables.

For the case in which A and B are functions of x and y only, it was proved in § 26 that there exist two independent integrals $u = c$, $v = k$ of (2). For the general case in which A and B, as well as C, are any functions of x, y, z, the same conclusion holds.*

* Picard, Traité d'Analyse, II, 1893, p. 298. To give a plausible argument (not a proof), note that, at an arbitrary point P: (x_0, y_0, z_0) in space, equations (2) determine a unique direction or straight line whose direction cosines are proportional to A (x_0, y_0, z_0), B, C. Consider a point P_1: (x_1, y_1, z_1) in this line at an infinitesimal distance from P_0. Equations (2) determine a unique direction at P_1 and hence determine a third point P_2 at an infinitesimal distance from P_1 and in the latter direction from it. In this manner we obtain a curve in space. By varying the initial point P_0, we obtain in all a doubly infinite system of integral curves of (2). The general one of these curves is determined by two equations involving x, y, z and two arbitrary constants c and k, whose solved forms are $u(x, y, z) = c$, $v = k$.

Any linear partial differential equation of the first order in n variables has $n-1$ independent solutions. Any function of them is a solution and every solution is a function of them.

32. **Three equations in three variables.** Consider

$$(3)\qquad \begin{aligned} Pf &\equiv A\frac{\partial f}{\partial x}+B\frac{\partial f}{\partial y}+C\frac{\partial f}{\partial z}=0,\\ Uf &\equiv D\frac{\partial f}{\partial x}+E\frac{\partial f}{\partial y}+F\frac{\partial f}{\partial z}=0,\\ Vf &\equiv G\frac{\partial f}{\partial x}+H\frac{\partial f}{\partial y}+I\frac{\partial f}{\partial z}=0, \end{aligned}$$

in which $A, \ldots, I$ are functions of x, y, z. Suppose they have a common solution $\Phi(x, y, z)$ not a constant. Then

$$(4)\qquad \frac{\partial \Phi}{\partial x}, \quad \frac{\partial \Phi}{\partial x}, \quad \frac{\partial \Phi}{\partial z}$$

are not all zero. When Φ is substituted for f, equations (3) become three linear homogeneous equations in the three unknowns (4). Since the latter are not all zero, the determinant

$$(5)\qquad \Delta \equiv \begin{vmatrix} A & B & C\\ D & E & F\\ G & H & I \end{vmatrix}$$

of their coefficients is identically zero.

Then there exist three functions l, m, n of x, y, z not all identically zero, such that

$$(6)\qquad lPf+mUf+nVf\equiv 0,$$

identically in x, y, z for all functions f, and the three equations (3) are said to be *linearly dependent*. For, if the minors

$$a=\begin{vmatrix} E & F\\ H & I \end{vmatrix}, \quad d=\begin{vmatrix} B & C\\ H & I \end{vmatrix}, \quad g=\begin{vmatrix} B & C\\ E & F \end{vmatrix}$$

of the elements of the first column in (5) are not all identically zero, we have (6) for $l=a$, $m=-d$, $n=g$. But if all nine minors of (5) are

identically zero and if $I \not\equiv 0$, for example, we see from $a \equiv 0$ and $b \equiv 0$, where b is the minor of B, that

$$E \equiv \frac{HF}{I}, \qquad D \equiv \frac{GF}{I}, \qquad Uf \equiv \frac{F}{I} Vf,$$

so that we have a linear dependence (6).

THEOREM 1. *If equations* (3) *have a common solution which is not a constant, they are linearly dependent.*

The problem of solving three linearly independent equations (3) is therefore trivial, since their only common solution is a constant. If the equations are dependent, so that (6) holds, where for example l is not identically zero, every common solution of $Uf = 0$ and $Vf = 0$ is a solution of $Pf = 0$, which may therefore be discarded. If four or more equations in three variables have a common solution which is not a constant, all but two of the equations are consequences of those two and may be discarded.

33. **Complete system of two equations.** In the commutator

$$(PU)f = P(Uf) - U(Pf), \tag{7}$$

of Pf with Uf, the second derivatives are seen to cancel as in § 21. Whence if we employ the notations (3), we readily obtain

$$(PU)f = (PD - UA)\frac{\partial f}{\partial x} + (PE - UB)\frac{\partial f}{\partial y} + (PF - UC)\frac{\partial f}{\partial z}, \tag{8}$$

so that the rule for forming the commutator is the same as that in § 21.

Let $Pf = 0$ and $Uf = 0$ be linearly independent and have a common solution Φ not a constant. Then, by (7), Φ is also a solution of $(PU)f = 0$. Hence, by Theorem 1, $Pf = 0$, $Uf = 0$, $(PU)f = 0$ are linearly dependent. In the linear relation between them, the coefficient of $(PU)f$ is not zero identically (since otherwise Pf and Uf would be linearly dependent) and may be divided out of the relation. This proves the following result.

THEOREM 2. *If $Pf = 0$ and $Uf = 0$ are linearly independent and have a common solution which is not a constant, then*

$$(PU)f = \varrho(x, y, z)\, Pf + \sigma(x, y, z)\, Uf, \tag{9}$$

identically in x, y, z and for every function f.

Two linearly independent equations $Pf = 0$ and $Uf = 0$ whose commutator is expressible linearly to terms of them, as in (9), are said to form a *complete system*. Hence Theorem 2 may be stated in the following form.

THEOREM 3. *If two linear partial differential equations of the first order are linearly independent and have a common solution which is not a constant, they form a complete system.*

In order to prove the converse of this theorem, we first show that if we replace a complete system $Pf = 0$, $Uf = 0$ by an equivalent system

$$(10)\quad Rf \equiv \alpha Pf + \beta Uf = 0, \quad Sf \equiv \gamma Pf + \delta Uf = 0, \quad \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \not\equiv 0,$$

in which $\alpha, \ldots, \delta$ are functions of x, y, z, we obtain a new complete system $Rf = 0$, $Sf = 0$. By definition, $(RS)f \equiv R(Sf) - S(Rf)$, whence

$$(RS)f \equiv R(\gamma Pf + \delta Uf) - S(\alpha Pf + \beta Uf).$$

In computing each of the four terms $R(\gamma Pf)$, etc., we must apply a linear differential operator R or S to a product of two functions. Hence we operate on each of the two factors in turn as in the differentiation of a product. First, if we operate each time on the Greek letters, we get

$$(11)\qquad (R\gamma - S\alpha)\, Pf + (R\delta - S\beta)\, Uf.$$

Next, if we operate on the Roman letters, we get

$$\gamma R(Pf) + \delta R(Uf) - \alpha S(Pf) - \beta S(Uf).$$

Inserting the expressions (10) for Rf and Sf, we get eight terms which, after cancellation of four in pairs, may be combined by (7) into

$$(12)\qquad (\alpha\delta - \beta\gamma) \cdot (PU)f.$$

Hence $(RS)f$ is equal to the sum of the expressions (11) and (12).

We assume that $Pf = 0$ and $Uf = 0$ form a complete system, so that (9) holds. Then (12), and hence also $(RS)f$, is a linear function of Pf and Uf. The latter are linear functions of Rf and Sf, as seen by solving relations (10). Hence

$$(13)\qquad (RS)f = \lambda(x, y, z)\, Rf + \mu(x, y, z)\, Sf.$$

Hence *if* $Pf = 0$ *and* $Uf = 0$ *form a complete system, any two linearly independent linear combinations* (10) *of them also form a complete system.*

Our next step is to prove that we can choose Rf and Sf in (10) so that $(RS)f \equiv 0$. Since $Pf = 0$ and $Uf = 0$, given by (3), are linearly independent, not all the three 2-rowed determinants of their coefficients are identically zero (since otherwise A, B, C would be proportional to D, E, F). Let, for example, $d = AE - BD \not\equiv 0$. Choose

$$Rf \equiv \frac{1}{d}(EPf - BUf) \equiv \frac{\partial f}{\partial x} + r\frac{\partial f}{\partial z}, \qquad r \equiv \frac{1}{d}(EC - BF),$$

$$Sf \equiv \frac{1}{d}(-DPf + AUF) \equiv \frac{\partial f}{\partial y} + s\frac{\partial f}{\partial z}, \qquad s \equiv \frac{1}{d}(AF - DC).$$

Then

$$(RS)f \equiv (Rs - Sr)\frac{\partial f}{\partial z}.$$

Comparing this with (13), we see that $\lambda \equiv 0$, $\mu \equiv 0$. Hence *any complete system is equivalent to a* JACOBIAN SYSTEM *whose commutator is identically zero.*

A Jacobian system has a common solution. For, let $u(x,y,z)$ and $v(x,y,z)$ be two independent solutions of $Rf = 0$. Then any function $F(u,v)$ of them is a solution. This F will be a solution also of $Sf = 0$, if (§ 29)

$$SF(u,v) \equiv \frac{\partial F}{\partial u}Su + \frac{\partial F}{\partial v}Sv \equiv 0. \tag{14}$$

But $(RS)f \equiv R(Sf) - S(Rf) \equiv 0$. Replacing f by u and v in turn, and recalling that $Ru \equiv 0$, $Rv \equiv 0$, we get $R(Su) \equiv 0$, $R(Sv) \equiv 0$. Thus Su and Sv are solutions of $Rf = 0$ and hence are functions $g(u,v)$ and $h(u,v)$ of u and v. Our equation (14) becomes

$$g(u,v)\frac{\partial F}{\partial u} + h(u,v)\frac{\partial F}{\partial v} \equiv 0,$$

which is known (§ 7) to have a solution $F(u,v)$.

This common solution F of $Rf = 0$ and $Sf = 0$ is a common solution of $Pf = 0$ and $Uf = 0$ in view of (10). This completes the proof of the following converse of Theorem 3.

THEOREM 4. *If $Pf = 0$ and $Uf = 0$ form a complete system, they have a common solution which is not a constant.*

34. **Complete system of r equations in n variables.** The preceding results may be extended readily by the same methods* to linear partial

* For a simple exposition of this classic theory, see L. Bianchi, Gruppi continui finiti di trasformazioni, 1918, 18–28; Goursat's Mathematical Analysis, vol. 2, part 2, p. 267 (English transl. by Hedrick and Dunkel).

differential equations $P_i f = 0$ of the first order in n variables. If n such equations have a common solution which is not a constant, they are linearly dependent. Hence consider $s\,(s<n)$ linearly independent equations $P_1 f = 0, \ldots, P_s f = 0$. If they have a common solution which is not a constant, each of the $1/2\,s(s-1)$ equations

$$(P_j P_k) f = 0 \qquad (j, k = 1, \ldots, s; j<k)$$

has this same solution. We therefore enlarge the given system of equations by annexing seriatim such of the new equations as are linearly independent of the former and those previously annexed. We form the commutators of the enlarged system and again annex seriatim those which are independent of the former and those previously annexed. This process must terminate since we saw that not more than $n-1$ independent equations have a common solution which is not a constant. Let the final system be

$$P_1 f = 0 \ldots . P_r f = 0 \qquad (r<n), \tag{15}$$

so that they are linearly independent, while the commutator of any pair of them is a linear function of $P_1 f, \ldots, P_r f$. Such a system is called a *complete system*. Hence all common solutions of any system are common solutions of a certain complete system.

Our complete system (15) is equivalent to a Jacobian system* $Q_1 f = 0, \ldots$ $\ldots, Q_r f = 0$, all of whose commutators are identically zero. The latter system, and hence also (15), has exactly $n-r$ independent solutions. For, by § 31, $Q_1 f = 0$ has $n-1$ independent solutions $u_1, \ldots, u_{n-1}$. In view of $(Q_1 Q_2) f = 0$, $Q_2 u_i$ is a function Φ_i of $u_1, \ldots, u_{n-1}$. Then $F(u_1, \ldots, u_{n-1})$ will be a solution also of $Q_2 f = 0$ if (§ 29)

$$Q_2 F \equiv \Phi_1 \frac{\partial F}{\partial u_1} + \cdots + \Phi_{n-1} \frac{\partial F}{\partial u_{n-1}} = 0.$$

By § 31, this equation in $n-1$ variables has $n-2$ independent solutions $v_1, \ldots, v_{n-2}$. In view of $(Q_1 Q_3) f = 0$ and $(Q_2 Q_3) f = 0$, we see that $Q_3 v_i$ is a solution of both $Q_1 f = 0$ and $Q_2 f = 0$ and hence is a function ψ_i of $v_1, \ldots, v_{n-2}$. Thus $\Phi(v_1, \ldots, v_{n-2})$ will be a solution of $Q_3 f = 0$ if

$$Q_3 \Phi \equiv \psi_1 \frac{\partial \Phi}{\partial v_1} + \cdots + \psi_{n-2} \frac{\partial \Phi}{\partial v_{n-2}} = 0.$$

* Obtained by solving (15) for r of the partial derivatives in terms of the remaining $n-r$.

This equation in $n-2$ variables has $n-3$ independent solutions which, being functions of $v_1, \ldots, v_{n-2}$, are solutions also of $Q_1 f = 0$ and $Q_2 f = 0$. We have now taken three of the r similar steps in the proof (by induction) of

THEOREM 5. *All common solutions of a complete system of r equations in n variables are functions of $n-r$ independent common solutions.*

35. **Standard methods of solving a complete system of two equations in three variables.** One method employs two independent solutions u and v of one of the equations, say $Uf = 0$. By Theorem 4 of § 33, there exists a function $F(u, v)$ which is a solution also of the other equation $Pf = 0$, whence

$$PF = \frac{\partial F}{\partial u} Pu + \frac{\partial F}{\partial v} Pv \equiv 0.$$

If u itself is not the desired common solution, then $Pu \not\equiv 0$ and

$$\frac{\partial F}{\partial u} + q \frac{\partial F}{\partial v} \equiv 0, \qquad q = \frac{Pv}{Pu}. \tag{16}$$

Since the partial derivatives of $F(u, v)$ are functions of u and v, the remaining quantity q in (16) must be a function of u and v. Hence (16) is an equation in u and v only, and is equivalent (§ 13) to an ordinary differential equation of the first order in two variables.

For example, the two equations

$$\left\{ \begin{aligned} Pf &\equiv xy^2 \frac{\partial f}{\partial x} - x^2 y \frac{\partial f}{\partial y} + \tfrac{1}{2}(y^2 - x^2) z \frac{\partial f}{\partial z} = 0, \\ Uf &\equiv \quad x \frac{\partial f}{\partial x} + y \quad \frac{\partial f}{\partial y} + \qquad\qquad z \frac{\partial f}{\partial z} = 0 \end{aligned} \right. \tag{17}$$

form a complete system since $(UP)f \equiv 2Pf$. Evident solutions of $Uf = 0$ are $u = y/x$, $v = z/x$. Then

$$Pu = -\frac{y}{x}(x^2 + y^2), \quad Pv = \frac{-z}{2x}(x^2 + y^2), \quad q = \frac{v}{2u}.$$

Thus (16) becomes

$$u \frac{\partial f}{\partial u} + \tfrac{1}{2} v \frac{\partial f}{\partial v} = 0, \quad \frac{du}{u} = \frac{2\,dv}{v}, \quad \log \frac{u}{v^2} = \text{const.}$$

Hence a common solution of (17) is u/v^2 or xy/z^2.

Another method reduces the solution of a complete system $Pf = 0$, $Uf = 0$, given by (3), to the finding of an integral $f = \text{const.}$ of the total differential equation

$$\begin{vmatrix} dx & dy & dz \\ A & B & C \\ D & E & F \end{vmatrix} = 0, \tag{18}$$

which follows from the given equations and

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = 0.$$

To find an integral $f = c$ of (18), note that for any chosen value of a the plane $z = x + ay$ intersects the desired integral surfaces $f = c$ in a family of curves whose differential equation is found by eliminating z and dz from (18) by means of

$$z = x + ay, \quad dz = dx + a\,dy, \tag{19}$$

and hence is of the form

$$g(x, y, a)\,dx + h(x, y, a)\,dy = 0.$$

For a arbitrary, let $\psi(x, y, a) = c$ be an integral of the latter. Eliminating a by means of $z = x + ay$, we obtain the desired integral

$$\psi\left(x, y, \frac{z-x}{y}\right) = c$$

of (18). The method requires modification (as in the example) when a does not actually occur in $\psi(x, y, a)$.

EXAMPLE. When the given equations are (17), we find that (18) is the product of $-\frac{1}{2}(x^2+y^2)$ by

$$yz\,dx + xz\,dy - 2xy\,dz = 0. \tag{20}$$

Inserting the values (19), we get

$$(ay - x)(y\,dx - x\,dy) = 0.$$

Since a drops out on removing the algebraic factor, the method fails and will fail when any homogeneous equation is used for the planes. But we shall succeed if we employ the equation $y = x + a$, $dy = dx$, instead of (19). Now (20) becomes

$$\frac{(2x+a)\,dx}{x(x+a)} = \frac{2\,dz}{z}, \quad d\log x(x+a) = d\log z^2, \quad \frac{x(x+a)}{z^2} = c.$$

Elimination of a gives $xy/z^2 = c$.

36. **Equation $Pf = 0$ determined by two solutions.** Two independent solutions $\Phi_1(x, y, z)$ and $\Phi_2(x, y, z)$ of

$$Pf \equiv A\frac{\partial f}{\partial x} + B\frac{\partial f}{\partial y} + C\frac{\partial f}{\partial z} = 0 \tag{21}$$

determine uniquely the ratios of A, B, C and hence determine the equation apart from a factor which is a function of x, y, z. For,

$$A\frac{\partial \Phi_1}{\partial x} + B\frac{\partial \Phi_1}{\partial y} + C\frac{\partial \Phi_1}{\partial z} \equiv 0, \qquad A\frac{\partial \Phi_2}{\partial x} + \cdots \equiv 0,$$

together with (21) require

$$\begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial \Phi_1}{\partial x} & \frac{\partial \Phi_1}{\partial y} & \frac{\partial \Phi_1}{\partial z} \\ \frac{\partial \Phi_2}{\partial x} & \frac{\partial \Phi_2}{\partial y} & \frac{\partial \Phi_2}{\partial z} \end{vmatrix} = 0, \tag{22}$$

whose expansion according to the elements of the first row is a linear partial differential equation* which has the solutions Φ_1 and Φ_2 and is uniquely determined by them.

37. **$Pf = 0$ invariant under a transformation.** By the simple method of § 29 we may express (21) in terms of new independent variables

$$x_1 = \lambda(x, y, z), \qquad y_1 = \mu(x, y, z), \qquad z_1 = \nu(x, y, z); \tag{23}$$

denote the resulting differential expression by $P_1 f$.

We shall say that $Pf = 0$ is invariant under the transformation (23) if

$$P_1 f \equiv \varrho(x_1, y_1, z_1)\left(A_1\frac{\partial f}{\partial x_1} + B_1\frac{\partial f}{\partial y_1} + C_1\frac{\partial f}{\partial z_1}\right), \tag{24}$$

where A_1, B_1, C_1 are the same functions of x_1, y_1, z_1 that A, B, C were of x, y, z. Since $\Phi_i(x, y, z)$ is a solution of (21), $\Phi_i(x_1, y_1, z_1)$ is then

* Not identically zero. For, if the determinant (22) be identically zero, a well known theorem states that f, Φ_1, Φ_2 are dependent functions. Since f is arbitrary, Φ_1 and Φ_2 would be dependent contrary to hypothesis.

a solution of $P_1 f = 0$. Since $P_1 f = Pf$ in view of relations (23), the latter solution becomes a solution of $Pf = 0$ when expressed in terms of x, y, z, and hence is a function of two independent solutions:

$$\Phi_i(x_1, y_1, z_1) = \psi_i[\Phi_1(x, y, z), \Phi_2(x, y, z)] \qquad (i = 1, 2). \tag{25}$$

Conversely, relations (25) imply that $P_1 f = 0$ has the solutions $\Phi_i(x_1, y_1, z_1)$, so that (24) holds by § 36.

Hence $Pf = 0$ *is invariant under the transformation* (23) *if and only if its independent solutions* Φ_i *satisfy relations* (25).

Let an infinitesimal transformation Uf generate a group of transformations (23) in which the coefficients are functions of a parameter t. Then $Pf = 0$ is invariant under the group if and only if (25) are consequences of (23), the coefficients of ψ_i being functions of t. By (15) of Chapter I,

$$\Phi_i(x_1, y_1, z_1) = \Phi_i + tU\Phi_i + \frac{t^2}{1\cdot 2} U(U\Phi_i) + \cdots \qquad (i = 1, 2), \tag{26}$$

for every t sufficiently small, where Φ_i denotes $\Phi_i(x, y, z)$.

If (25) hold, the coefficient of t in the expansion of its second member must be equal to $U\Phi_i$ by (26), whence*

$$U\Phi_1 = F_1(\Phi_1, \Phi_2), \qquad U\Phi_2 = F_2(\Phi_1, \Phi_2). \tag{27}$$

Conversely, (27) imply that

$$U(U\Phi_i) = UF_i(\Phi_1, \Phi_2) = \frac{\partial F_i}{\partial \Phi_1} U\Phi_1 + \frac{\partial F_i}{\partial \Phi_2} U\Phi_2 = \frac{\partial F_i}{\partial \Phi_1} F_1 + \frac{\partial F_i}{\partial \Phi_2} F_2,$$

and similarly that $U[U(U\Phi_i)]$ is a function of Φ_1, Φ_2. Then from (26) we obtain relations of the type (25).

THEOREM 6. *An equation* $Pf = 0$ *having the independent solutions* Φ_1 *and* Φ_2 *is invariant under (the group generated by)* Uf *if and only if* $U\Phi_1$ *and* $U\Phi_2$ *are functions of* Φ_1 *and* Φ_2 *as in* (27).

This criterion for the invariance of $Pf = 0$ under Uf involves the solutions Φ_i which we desire to find. Hence we seek a criterion not involving the Φ_i. First, let (27) hold. We employ the commutator

$$(UP)\Phi_i = U(P\Phi_i) - P(U\Phi_i) = U(0) - P[F_i(\Phi_1, \Phi_2)] = 0.$$

* These follow also from (13) of Chapter I.

Hence $(UP)f = 0$ has the same solutions Φ_i as $Pf = 0$, whence (§ 36)

$$(UP)f = \varrho(x, y, z)\, Pf. \tag{28}$$

Conversely, the latter implies that the solution Φ_i of $Pf = 0$ is a solution of $(UP)f = 0$, whence $P(U\Phi_i) = 0$, so that (27) follow.

THEOREM 7. *$Pf = 0$ is invariant under Uf if and only if the commutator $(UP)f$ is the product of Pf by a function of x, y, z.*

In the example (17), $(UP)f = 2\,Pf$. Hence $Pf = 0$ is invariant under Uf.

38. **Solution of equation $Pf = 0$ invariant under Uf.** First let Pf and Uf be linearly independent. Then, by (28), $Pf = 0$ and $Uf = 0$ form a complete system. Suppose we have found by one of the methods in § 35 a common solution $\Phi = \Phi(x, y, z)$ which is not a constant. Then a second solution can be found by quadratures as follows. By permuting x, y, z if necessary, we may assume that z actually occurs in Φ. Then we may introduce the new variables x, y, Φ in place of x, y, z. By § 29, Pf and Uf become, when expressed in terms of x, y, Φ,

$$P_1 f = A_1 \frac{\partial f}{\partial x} + B_1 \frac{\partial f}{\partial y}, \qquad U_1 f = \xi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y}, \tag{29}$$

since the coefficients of $\partial f / \partial \Phi$ are $P\Phi = 0$, and $U\Phi = 0$. Here A_1, B_1, ξ_1, η_1 are the functions A, B, ξ, η expressed in terms of x, y, Φ. By (28),

$$(U_1 P_1)f = (UP)f = \varrho\, Pf = \varrho_1 P_1 f.$$

so that $P_1 f = 0$ is invariant under $U_1 f$ and

$$\frac{B_1\, dx - A_1\, dy}{A_1 \eta_1 - B_1 \xi_1} \tag{30}$$

is an exact differential (§ 15), in whose integration by quadratures Φ plays the rôle of a constant. Note that the denominator is not zero identically since otherwise $P_1 f$ and $U_1 f$ would be linearly dependent, contrary to the linear independence of Pf and Uf.

EXAMPLE. Let Pf and Uf be given by (17). We found above the common solution $\Phi = xy/z^2$. Then

$$P_1 f = xy^2 \frac{\partial f}{\partial x} - x^2 y \frac{\partial f}{\partial y}, \qquad U_1 f = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y},$$

$$\int \frac{-x^2 y\, dx - xy^2 dy}{xy\,(x^2 + y^2)} = -\tfrac{1}{2} \log (x^2 + y^2). \tag{30'}$$

Hence $Pf = 0$ has the independent solutions xy/z^2 and $x^2 + y^2$.

Second, let $Uf \equiv vPf$. We have

$$(vP, V)f \equiv vP(Vf) - V(vPf) \equiv vP(Vf) - Vv \cdot Pf - vV(Pf),$$

$$(vP, V)f \equiv v(PV)f - Vv \cdot Pf. \tag{31}$$

The case $V \equiv P$ gives $(vP, P)f \equiv -Pv \cdot Pf$. Hence, by Theorem 7, $Pf = 0$ is invariant under $Uf \equiv vPf$ for every function v of x, y, z. Since no new information is gained concerning a particular equation $Pf = 0$ from the fact that it is invariant under $Uf \equiv vPf$, that infinitesimal transformation should not be expected to aid in the solution of $Pf = 0$. We now have $U_1 f \equiv vP_1 f$, so that the denominator in (30) is identically zero, and the fact that $P_1 f = 0$ is invariant under $U_1 f$ is of no aid in its integration (cf. § 15).

39. **Jacobi's identity.** If Uf, Vf, Pf are three linear partial differential expressions of the first order in any number of variables, then

$$((UV)P)f + ((VP)U)f + ((PU)V)f \equiv 0. \tag{32}$$

Since

$$(UV)f \equiv U(Vf) - V(Uf),$$

$$((UV)P)f \equiv UVPf - VUPf - PUVf + PVUf,$$

where $UVPf$ means $U[V(Pf)]$. Permuting the letters U, V, P cyclically, we get

$$((VP)U)f \equiv VPUf - PVUf - UVPf + UPVf,$$

$$((PU)V)f \equiv PUVf - UPVf - VPUf + VUPf.$$

By adding the three relations we get (32).

40. **Equation invariant under two infinitesimal transformations** We shall first prove the existence of two infinitesimal transformations Uf and Vf each leaving invariant a given partial differential equation $Pf = 0$ in three variables (and such that Uf, Vf, Pf are linearly independent). For, by § 31, $Pf = 0$ has two independent solutions $u(x, y, z)$ and $v(x, y, z)$, which are therefore independent with respect to two of the variables, say x and y. In terms of the new variables $x_1 = u$, $y_1 = v$, $z_1 = z$, the equation $Pf = 0$ becomes (§ 29)

$$Px_1 \cdot \frac{\partial f}{\partial x_1} + Py_1 \cdot \frac{\partial f}{\partial y_1} + Pz_1 \cdot \frac{\partial f}{\partial z_1} = 0 \quad \text{or} \quad \frac{\partial f}{\partial z_1} = 0.$$

By inspection, the latter is invariant under

$$\frac{\partial f}{\partial x_1}, \qquad \frac{\partial f}{\partial y_1}.$$

These become the desired Uf and Vf when we return to the initial variables.

We next prove the following fundamental result.

THEOREM 8. *If the linear partial differential equation $Pf = 0$ is invariant under two infinitesimal transformations Uf and Vf, it is invariant also under their commutator $(UV)f$.*

By Theorem 7,

$$(UP)f = \varrho Pf, \qquad (VP)f = \sigma Pf, \tag{33}$$

where ϱ and σ are functions of x, y, z. Also,

$$(PU)f = -(UP)f.$$

Hence (32) gives

$$((UV)P)f = (\varrho P, V)f - (\sigma P, U)f.$$

By (31),

$$(\varrho P, V)f = \varrho(PV)f - V\varrho \cdot Pf, \qquad (\sigma P, U)f = \sigma(PU)f - U\sigma \cdot Pf.$$

Applying also (33), we get

$$((UV)P)f = \tau Pf, \qquad \tau = U\sigma - V\varrho. \tag{34}$$

By Theorem 7, Pf is invariant under $(UV)f$.

41. **Solution of $Pf = 0$ invariant under Uf and Vf when P, U, V are linearly independent.** We employ the notations (3) and (5). Then Δ is not identically zero since P, U, V are linearly independent (§ 32). Hence we can solve relations (3) for the three partial derivatives as linear functions of P, U, V. Substituting the resulting values in the expression for the commutator $(UV)f$ as a linear function of those partial derivatives, we obtain a relation of the form

$$(UV)f = \lambda Uf + \mu Vf + \nu Pf, \tag{35}$$

in which λ, μ, ν are known functions of x, y, z. Since $Pf = 0$ is invariant under Uf and Vf, we have (34). Insert the expression (35) for (UV) and apply (31) and (33). We get

$$\begin{aligned}(\lambda Uf + \mu Vf + \nu Pf, P)f &= \lambda(UP)f - P\lambda \cdot Uf + \mu(VP)f - P\mu \cdot Vf - P\nu \cdot Pf \\ &= (\lambda\varrho + \mu\sigma - P\nu)Pf - P\lambda \cdot Uf - P\mu \cdot Vf.\end{aligned}$$

This must be identical with τPf by (34). But P, U, V are linearly independent. Hence $P\lambda \equiv 0$, $P\mu \equiv 0$, so that λ and μ are solutions of $Pf = 0$, which is invariant under Uf, Vf. Thus, by (27),

$$\lambda, \quad \mu, \quad U\lambda, \quad U\mu, \quad V\lambda, \quad V\mu \tag{36}$$

are all solutions of $Pf = 0$. If any two of these six functions are independent, we have found the complete solution of $Pf = 0$ by algebraic processes and differentiations.

There remain two cases. Either λ and μ are both constants, a case treated under (ii); or else λ (for example) is not a constant and the remaining five functions (36) are all functions of λ. The latter case will be treated first.

(i) Let λ be a solution, not a constant, of $Pf = 0$, such that $U\lambda = g(\lambda)$, $V\lambda = h(\lambda)$. Then the infinitesimal transformation*

$$Wf \equiv h(\lambda) \cdot Uf - g(\lambda) \cdot Vf$$

leaves $Pf = 0$ invariant. For, by (31),

$$(hU, P)f \equiv h(UP)f - Ph \cdot Uf, \qquad (gV, P)f \equiv g(VP)f - Pg \cdot Vf,$$

and $Ph \equiv Pg \equiv 0$, since $P\lambda \equiv 0$. Hence, by (33),

$$(WP)f \equiv h(UP)f - g \cdot (VP)f \equiv (h\varrho - g\sigma)Pf.$$

Furthermore, $W\lambda = hg - gh \equiv 0$. Hence we may apply § 38 (with W in place of U) and find by quadratures a common solution, in addition to λ, of the complete system $Pf = 0$, $Wf = 0$.

For example, the equation

$$Pf \equiv (x - y - z + 2)\frac{\partial f}{\partial x} + 2(y - x + z)\frac{\partial f}{\partial y} + (x - y - z)\frac{\partial f}{\partial z} = 0$$

is invariant under the two infinitesimal transformations

$$Uf \equiv \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}, \qquad Vf \equiv (y + 2z + 1)\left(\frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\right),$$

since $(UP)f \equiv 0$, $(VP)f \equiv 0$. Also, Pf, Uf, and Vf are linearly independent since the determinant of their coefficients is

$$4(y + 2z + 1)(y - x + z - 1).$$

* If $Wf \equiv 0$, then $h(\lambda) \equiv 0$, $g(\lambda) \equiv 0$, since Uf and Vf are linearly independent by hypothesis. Then λ is a common solution of $Pf = 0$, $Uf = 0$, $Vf = 0$, contrary to Theorem 1.

We find that $(UV)f \equiv k\,Vf$, $k = 2/(y+2z+1)$. Hence k, and also $\lambda = y+2z$, is a solution of $Pf = 0$. Also, $U\lambda \equiv 2$, $V\lambda \equiv 0$. Hence the problem falls under the present case (i) with $Wf \equiv -2\,Vf$. We remove a factor and employ

$$Wf \equiv \frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}, \qquad (WP)f \equiv 0.$$

To proceed as in § 38 (with W in place of U), we introduce the new variables x, y, $\Phi = y+2z$, and find that Pf and Wf become

$$P_1 f \equiv (x - \tfrac{1}{2}y - \tfrac{1}{2}\Phi + 2)\frac{\partial f}{\partial x} + (y - 2x + \Phi)\frac{\partial f}{\partial y}, \qquad W_1 f \equiv \frac{\partial f}{\partial x} + 2\frac{\partial f}{\partial y}.$$

By (30),

$$\frac{(y-2x+\Phi)\,dx - (x - \tfrac{1}{2}y - \tfrac{1}{2}\Phi + 2)\,dy}{2y - 4x + 2\Phi - 4} = 0$$

is an exact differential. In the integration we regard Φ as a constant, write t for $x - \tfrac{1}{2}y - \tfrac{1}{2}\Phi$, and eliminate x. We get

$$\frac{t\,dt}{2(t+1)} + \tfrac{1}{2}dy = 0, \qquad t - \log(t+1) + y = c.$$

Returning to the initial variables, we obtain the second solution

$$x - z - \log(x - y - z + 1).$$

(ii) Let λ and μ in (35) be constants. Without loss of generality we may take $\mu = 0$. For, if $\mu \neq 0$, $Pf = 0$ is invariant under

$$U_1 f = \frac{\lambda}{\mu}Uf + Vf, \qquad V_1 f = -Uf,$$

for which, by (35),

$$(U_1 V_1)f = -(VU)f = (UV)f = \mu\left(\frac{\lambda}{\mu}Uf + Vf\right) + \nu Pf,$$
$$= \mu\, U_1 f + \nu Pf,$$

and this is of the form (35) with $\mu = 0$. Hence we may write

$$(UV)f \equiv c\,Uf + \nu(x, y, z)\,Pf \qquad (c \text{ a constant}). \tag{37}$$

Since $Pf = 0$ is invariant under Uf and Vf, we have

$$(UP)f \equiv \varrho\,Pf, \qquad (VP)f \equiv \sigma\,Pf. \tag{38}$$

The first relation shows that $Pf = 0$ and $Uf = 0$ form a complete system; let Φ be their unknown common solution. The second relation and (37) may be written in the forms

$$V(Pf) - P(Vf) \equiv \sigma\,Pf, \qquad U(Vf) - V(Uf) \equiv c\,Uf + \nu\,Pf.$$

Take $f = \Phi$ and apply $P\Phi \equiv 0$, $U\Phi \equiv 0$. We see that $P(V\Phi) \equiv 0$, $U(V\Phi) \equiv 0$, so that $V\Phi$ is a solution of our complete system $Pf = 0$, $Uf = 0$, and hence is a function of Φ alone, $V\Phi \equiv \psi(\Phi)$. If $\psi(\Phi)$ were identically zero, $Vf = 0$, $Pf = 0$, and $Uf = 0$ would have a common solution Φ, not a constant, and hence by Theorem 1 would be linearly dependent, contrary to hypothesis. Hence $\psi(\Phi) \not\equiv 0$. As in § 15, we can therefore determine a function $\chi(\Phi)$ such that $V\chi \equiv 1$. Hence there exists a function χ for which

$$P\chi \equiv 0, \qquad U\chi \equiv 0, \qquad V\chi \equiv 1. \tag{39}$$

Employing the expressions (3) for Pf, Uf, Vf, and recalling that the determinant Δ is not identically zero, we see that (39) can be solved as linear equations for

$$\frac{\partial \chi}{\partial x}, \qquad \frac{\partial \chi}{\partial y}, \qquad \frac{\partial \chi}{\partial z}, \tag{40}$$

and their values substituted in

$$d\chi = \frac{\partial \chi}{\partial x}\,dx + \frac{\partial \chi}{\partial y}\,dy + \frac{\partial \chi}{\partial z}\,dz.$$

But the desired expression for $d\chi$ may be found more simply by a device. Since

$$P\chi \equiv 0, \quad U\chi \equiv 0, \quad V\chi \cdot d\chi - \left(\frac{\partial \chi}{\partial x}\,dx + \frac{\partial \chi}{\partial y}\,dy + \frac{\partial \chi}{\partial z}\,dz\right) \equiv 0$$

are homogeneous in (40), the determinant of their coefficients must be identically zero:

$$\begin{vmatrix} A & B & C \\ D & E & F \\ G\,d\chi - dx & H\,d\chi - dy & I\,d\chi - dz \end{vmatrix} \equiv 0.$$

Employing the determinant Δ in (5), we get at once

$$\Delta\,d\chi - \begin{vmatrix} A & B & C \\ D & E & F \\ dx & dy & dz \end{vmatrix} \equiv 0.$$

Hence

$$\chi = \int \frac{\begin{vmatrix} dx & dy & dz \\ A & B & C \\ D & E & F \end{vmatrix}}{\Delta}.$$

If $c = 0$ in (37), we see from it and (38) that $Pf = 0$ and $Vf = 0$ form a complete system having a common solution ψ for which $P(U\psi) = 0$, $V(U\psi) = 0$, whence $U\psi$ is a function of ψ alone. Without loss of generality (§ 15), we may take $U\psi = -1$. As before

$$\psi = \int \frac{\begin{vmatrix} dx & dy & dz \\ A & B & C \\ G & H & I \end{vmatrix}}{\Delta}.$$

Since $U\chi = 0$, ψ is independent of χ.

If $c \neq 0$, it remains to find a solution of $Pf = 0$ independent of the solution χ. Since χ is a common solution of $Pf = 0$ and $Uf = 0$, we may proceed as in § 38 and find by quadrature the desired new solution.

THEOREM. *If $Pf = 0$ is invariant under two given infinitesimal transformations Uf and Vf, such that the latter and Pf are linearly independent, the integration of $Pf = 0$ requires only quadratures.*

As an example under case (ii), the equation

$$Pf = 2x\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} - (x+2z)\frac{\partial f}{\partial z} = 0 \tag{41}$$

is invariant under

$$Uf = (x+4z)\frac{\partial f}{\partial z}, \qquad Vf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z},$$

since $(UP)f = 0$, $(VP)f = 0$. Here $(UV)f = 0$ and

$$\Delta = x(x+4z)(2y-x),$$

so that our problem falls under case (ii). Since

$$\begin{vmatrix} dx & dy & dz \\ 2x & x & -x-2z \\ 0 & 0 & x+4z \end{vmatrix} = x(x+4z)(dx-2dy),$$

a common solution of $Pf = 0$ and $Uf = 0$ is

$$\chi = \int \frac{dx-2dy}{2y-x} = -\log(x-2y).$$

While we might compute the integral ϕ, it is simpler to proceed as suggested under the case $c \neq 0$ and employ the new variables x, $w = x - 2y$, z. Then $Pf = 0$ becomes

$$2x \frac{\partial f}{\partial x} - (x+2z) \frac{\partial f}{\partial z} = 0,$$

which is invariant under Uf. From the resulting integrating factor, we conclude that

$$\frac{(x+2z)\,dx + 2x\,dz}{2x(x+4z)} = 0$$

is an exact differential; in fact, of $\frac{1}{4} \log (x^2 + 4xz)$. Hence $x - 2y$ and $x^2 + 4xz$ are two independent solutions of $Pf = 0$.

Further examples of greater intrinsic interest are solved by the method of this case (ii) in § 49. Two instructive, but longer problems of geometrical origin are treated by this method by Lie-Scheffers, Differentialgleichungen, pp. 453–456.

42. **Solution of $Pf = 0$ invariant under Uf and Vf when P, U, V are linearly dependent.** It is understood (end of § 38) that Uf is not the product of Pf by a function of x, y, z. But Pf, Uf, Vf are by hypothesis connected by a linear relation. The coefficient of Vf is not identically zero and may be divided out, giving

$$Vf = \Phi Uf + \nu Pf. \tag{42}$$

Since $Pf = 0$ is invariant under Uf and Vf,

$$(UP)f = \varrho Pf, \qquad (VP)f = \sigma Pf.$$

Hence by (31),

$$\sigma Pf = (\Phi U + \nu P, P)f = \Phi(UP)f - P\Phi \cdot Uf - P\nu \cdot Pf$$
$$= (\Phi\varrho - P\nu)\,Pf - P\Phi \cdot Uf.$$

Thus $P\Phi = 0$, since otherwise Uf would be a multiple of Pf. Hence we know in advance the solution Φ of $Pf = 0$. We may exclude the case in which Φ is a constant, since Vf in (42) is then not regarded as essentially distinct from Uf with respect to $Pf = 0$. In fact, when $Pf = 0$ is invariant under Uf, it is necessarily invariant under $\Phi Uf + \nu Pf$, for every constant Φ and function ν, since (as just shown)

$$(\Phi U + \nu P, P)f = (\Phi\varrho - P\nu)\,Pf,$$

and no new knowledge is added by the assumption that $Pf = 0$ is invariant also under (42).

We seek a solution of $Pf = 0$ which shall be independent of the known solution Φ. Since $(UP)f = \varrho\, Pf$, it is furnished by $U\Phi$ if $U\Phi$ is independent of Φ. Henceforth, let $U\Phi$ be a function of Φ.

For definiteness, let the variable z actually occur in Φ, and introduce x, y, Φ as new variables. Since $P\Phi = 0$, Pf and Uf become

$$P_1 f = Px\frac{\partial f}{\partial x} + Py\frac{\partial f}{\partial y}, \qquad U_1 f = Ux\frac{\partial f}{\partial x} + Uy\frac{\partial f}{\partial y} + U\Phi\frac{\partial f}{\partial \Phi}.$$

If $U\Phi = 0$, $U_1 f$ gives us an integrating factor of the ordinary differential equation corresponding to $P_1 f = 0$. But if $U\Phi \not\equiv 0$, $U_1 f$ alters Φ, which occurs only as a parameter in $P_1 f = 0$ and is evidently* of no assistance in the integration of $P_1 f = 0$. We must therefore solve the corresponding ordinary differential equation in x and y without any effective assistance from the known infinitesimal transformations.

THEOREM. *If $Pf = 0$ is invariant under two essentially distinct infinitesimal transformations Uf and Vf such that the latter and Pf are linearly dependent, one solution Φ is known in advance from* (42) *and the determination of a second solution requires, when $U\Phi$ is a function of Φ, the integration of an ordinary differential equation of the first order in two variables.*

To give a brief, but highly artificial, illustration, note that the equation (41) is invariant also under

$$Wf = \frac{x+4z}{xy+xz-y^2}\,\frac{\partial f}{\partial z}.$$

Since $Uf = \Phi\, Wf$, where $\Phi = xy+xz-y^2$, Φ is a solution of $Pf = 0$. Then $U\Phi = x^2+4xz = \psi$ is a new solution. From them we obtain $\psi - 4\Phi = (x-2y)^2$, and hence have our former solutions.

CHAPTER IV.

Ordinary differential equations of the second order.

The primary object of this chapter is to show how to utilize the knowledge that a differential equation of the second order is invariant under one or two known infinitesimal transformations in order to simplify the work of its integration. In the first case it is necessary to solve auxiliary differential equations of the first order, but in the second case we succeed by quadratures alone. The methods may be readily extended to differential equations of higher orders.

* For example, $A(x, y)\,\partial f/\partial x + B(x, y)\,\partial f/\partial y = 0$ is invariant under $\partial f/\partial z$ whatever be A and B. Since the invariance implies no information concerning A and B, it cannot aid in solving the equation.

The important secondary objects of the chapter are the development of a simple theory of differential invariants, an introduction to the theory of closed system of infinitesimal transformations, including the four canonical types of the systems determined by two infinitesimal transformations, and the determination of the number of linearly independent infinitesimal transformations which leave a differential equation invariant.

43. **Second extension of an infinitesimal transformation** Uf. We may best utilize the method employed in § 23 to find the first extension $U'f$ if we do not employ the explicit fractional expression for y_1', but use the general notation

$$x_1 = \Phi(x, y, a), \qquad y_1' = \chi(x, y, y', a). \tag{1}$$

Then $y'' = dy'/dx$ is transformed into

$$y_1'' = \frac{dy_1'}{dx_1} = \frac{d\chi}{d\Phi} = \frac{\chi_x + \chi_y y' + \chi_{y'} y''}{\Phi_x + \Phi_y y'}.$$

Denote the final fraction by N/D. We desire the value of its derivative with respect to a for $a = a_0$. For $a = a_0$, (1) reduces to the identity transformation $x_1 = x$, $y_1' = y'$. Hence

$$\Phi_x = 1, \ \Phi_y = 0, \ \chi_x = 0, \ \chi_y = 0, \ \chi_{y'} = 1, \ N = y'', \ D = 1, \text{ when } a = a_0.$$

Since $d\chi/da$ reduces to η' and $d\Phi/da$ to ξ by § 23, we have

$$\left(\frac{dN}{da}\right)_{a=a_0} = \eta_x' + \eta_y' y' + \eta_{y'}' y'' = \frac{d\eta'}{dx}, \qquad \left(\frac{dD}{da}\right)_{a=a_0} = \xi_x + \xi_y y' = \frac{d\xi}{dx}.$$

Hence

$$\eta'' = \frac{d}{da}\left(\frac{N}{D}\right)_{a=a_0} = \frac{d\eta'}{dx} - y'' \frac{d\xi}{dx}.$$

The second extension of Uf is therefore

$$U''f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'} + \eta'' \frac{\partial f}{\partial y''}, \tag{2}$$

where

$$\eta' = \frac{d\eta}{dx} - y' \frac{d\xi}{dx} = \eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2, \tag{3}$$

$$\eta'' = \frac{d\eta'}{dx} - y'' \frac{d\xi}{dx} = \eta_x' + \eta_y' y' + (\eta_{y'}' - \xi_x - \xi_y y') y''. \tag{4}$$

By an argument analogous to that in § 28, we obtain the

THEOREM. *The differential equation $F(x, y, y', y'') = 0$ of the second order is invariant under the infinitesimal transformation Uf if and only if the ordinary equation $F = 0$ in four independent variables x, y, y', y'' is invariant under the second extension $U''f$ of Uf, and hence if $U''F$ vanishes for all sets of values of the four variables satisfying $F = 0$.*

For example, the first extension of

$$Uf = ny\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}$$

was found at the end of § 26. Its second extension is

$$U''f \equiv ny\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} + (1 - ny'^2)\frac{\partial f}{\partial y'} - 3ny'y''\frac{\partial f}{\partial y''}.$$

Since $U''y'' = -3ny'y''$ is zero when $y'' = 0$, the differential equation $y'' = 0$ is invariant under Uf. This is geometrically evident when $n = -1$, since Uf is then the infinitesimal rotation about the origin, while the system of all straight lines in the plane (the integral curves of $y'' = 0$) is invariant under all rotations.

44. **Differential invariants of the second order of Uf.** By § 26, there exists an ordinary invariant $u(x, y)$ of Uf, i. e., a solution of $Uf = 0$, and a differential invariant $v(x, y, y')$ of the first order, i. e., a solution involving y' of $U'f = 0$, where $U'f$ is the first extension of Uf. But it is not necessary to proceed by a similar process of integration in order to obtain a differential invariant w of the second order of Uf, i. e., a solution involving y'' of $U''f = 0$, where $U''f$ is the second extension of Uf. In fact, Lie gave the following device to find w by differentiation.

If a and b are any constants,

$$v(x, y, y') - au(x, y) - b = 0 \tag{5}$$

is a differential equation invariant under Uf since its left member is a differential invariant. Keeping a fixed, but making b vary, we obtain an infinitude of differential equations, each invariant under Uf. Thus each has a family of integral curves which are permuted by Uf. The totality of the curves of the infinitude of families is therefore invariant under Uf. This totality of curves is the set of integral curves of a differential equation of the second order which is invariant under Uf and is obtained by differentiating (5), since we obtain a result lacking b and hence true for all the curves in question. We get

$$dv - a\,du = 0$$

or $w-a=0$, if we employ the abbreviation

$$(6) \qquad w=\frac{dv}{du}=\frac{\dfrac{dv}{dx}}{\dfrac{du}{dx}}=\frac{\dfrac{\partial v}{\partial x}+\dfrac{\partial v}{\partial y}y'+\dfrac{\partial v}{\partial y'}y''}{\dfrac{\partial u}{\partial x}+\dfrac{\partial u}{\partial y}y'}.$$

Since $w-a=0$ is invariant under Uf, we conclude from § 43 that

$$U''(w-a)=U''w$$

vanishes for all sets of solutions of $w-a=0$. But $U''w$ does not involve the arbitrary constant a which appears in $w-a=0$. Hence the preceding condition requires that $U''w$ be identically zero.*

Hence w is invariant under $U''f$. Moreover, the coefficient of y'' in (6) is not identically zero since y' occurs in v by hypothesis. Hence w is a differential invariant of the second order of Uf.

THEOREM. *Given an ordinary invariant $u(x, y)$ and a differential invariant $v(x, y, y')$ of the first order of an infinitesimal transformation Uf, we obtain by differentiation a differential invariant $w=dv/du$ of the second order of Uf.*

Similarly, differential invariants of higher orders are furnished by further differentiations:

$$\frac{d^2v}{du^2}, \qquad \frac{d^3v}{du^3}, \qquad \cdots$$

45. Integration of all differential equations of the second order invariant under Uf. By an argument like that in § 30, we see from the last theorem that the most general differential equation of the second order invariant under Uf is obtained by equating to zero a function of u, v, w and hence its solved form is

$$(7) \qquad \frac{dv}{du}=F(u, v).$$

* While this is rather evident, a formal proof is readily supplied. The expression (6) for w shows that it is a linear function $ky''+l$ of y''. In the symbol (2) for $U''f$, η'' is given by (4) and is linear in y'', so that $U''w$ is a linear function $cy''+d$ of y''. By means of $w-a=0$, we eliminate y'' from $U''w=0$ and obtain

$$c\left(\frac{a-l}{k}\right)+d=0.$$

This must hold for every a (as well as every x, y, y'). Hence $c=0$ and then $d=0$. In other words, $U''w$ is identically zero.

Suppose we have integrated this differential equation of the first order in the variables u, v and found an integral $\Phi(u, v) = c$. The latter is a differential equation of the first order in x, y which is invariant under Uf (since u and v are invariant), whence an integrating factor is given by § 15.

EXAMPLE 1. Let Uf be $y\,\partial f/\partial y$, so that $\xi = 0$, $\eta = y$. Then $\eta' = y'$,

$$U'f = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'}.$$

Evident solutions of $U'f = 0$ are $u = x$, $v = y'/y$. Then

$$w = \frac{dv}{du} = \frac{d}{dx}\left(\frac{y'}{y}\right) = \frac{yy'' - y'^2}{y^2}, \qquad w = \frac{y''}{y} - v^2.$$

Hence every differential equation of the second order invariant under $y\,\partial f/\partial y$ is of the form

$$\frac{y''}{y} + \phi\left(x, \frac{y'}{y}\right) = 0.$$

An interesting case is that in which ϕ is a linear function $Q(x) + N(x)\,y'/y$ of y'/y. Then we obtain the homogeneous linear differential equation

$$y'' + N(x)\,y' + Q(x)\,y = 0. \tag{8}$$

The first step in our method of integration consists in reducing (8) to the type (7) by the substitution $x = u$, $y'/y = v$, which imply

$$\frac{y''}{y} = \frac{dv}{du} + v^2$$

by the above relations. Thus (8) becomes a Riccati equation

$$\frac{dv}{du} + v^2 + N(u)\,v + Q(u) = 0.$$

Let $v_0(u)$ be a particular solution und write $v = 1/z + v_0$, where $z = z(u)$ is the new dependent variable. Then

$$\frac{1}{z^2}\frac{dz}{du} = \frac{dv_0}{du} - \frac{dv}{du} = v^2 - v_0^2 + (v - v_0)N = \frac{1}{z}\left(\frac{1}{z} + 2v_0 + N\right),$$

$$\frac{dz}{du} = (N + 2v_0)z + 1,$$

which is a linear differential equation solved in § 16.

EXAMPLE 2. Any linear differential equation of the second order

$$y'' + N(x)\,y' + Q(x)\,y + R(x) = 0 \tag{9}$$

can be integrated by quadratures when there is known a particular integral $z(x)$ of the corresponding abridged equation (8). For, if y is any integral of (9), $y + cz$ is evidently

an integral when c is any constant. In other words, (9) is invariant under the group of transformations $x_1 = x$, $y_1 = y + cz$, which is generated by the infinitesimal transformation

$$Uf = z(x)\frac{\partial f}{\partial y}.$$

In order to apply our general theory, we need in addition to the evident invariant $u = x$ of Uf also a differential invariant v of the first order, i. e., a solution of

$$Uf = z(x)\frac{\partial f}{\partial y} + z'(x)\frac{\partial f}{\partial y'} = 0.$$

Hence v is an integral of the corresponding ordinary differential equation

$$\frac{dy}{z(x)} = \frac{dy'}{z'(x)}, \qquad z\,dy' - z'\,dy = 0.$$

We may therefore take $v = zy' - z'y$. Then

$$w = \frac{dv}{du} = \frac{z\,dy' + y'z'\,dx - (z'dy + yz''dx)}{dx} = zy'' - yz''$$

is a differential invariant of Uf of the second order. By the general theory we know that our equation (9) can be given the form (7), viz.,

$$zy'' - yz'' = F(x, zy' - z'y).$$

To obtain the latter we recall that z is an integral of (8):

$$z'' + N(x)z' + Q(x)z = 0.$$

Hence we multiply the latter by $-y$, and (9) by z, and add. We get

$$zy'' - yz'' + (zy' - z'y)N + zR = 0,$$

or

$$\frac{dv}{dx} + vN + zR = 0.$$

By § 16, an integral of this linear differential equation of the first order is

$$v = zy' - z'y = e^{-\int N dx}\left(-\int zRe^{\int N dx}\,dx + c\right).$$

Denote the right member by $\Phi(x)$. Then

$$d\left(\frac{y}{z}\right) = \frac{\Phi(x)\,dx}{z^2}, \qquad y = z\left(\int \frac{\Phi(x)\,dx}{z^2} + \text{const.}\right).$$

By the method used in these examples we obtain the

TABLE OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER INVARIANT UNDER THE ACCOMPANYING INFINITESIMAL TRANSFORMATION.

$$y'' = F(ax+by, y'), \qquad b\frac{\partial f}{\partial x} - a\frac{\partial f}{\partial y}.$$

$$\frac{y''}{x^{n-2}} = F\left(\frac{y}{x^n}, \frac{y'}{x^{n-1}}\right), \qquad x\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y}.$$

$$\frac{y''}{(1+y'^2)^{\frac{3}{2}}} = F\left(x^2+y^2, \frac{y-xy'}{x+yy'}\right), \qquad -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

$$y''x^{n+2} + (1-n)x^{n+1}y' = F\left(\frac{y}{x^n}, xy'-ny\right), \qquad x^n\left(x\frac{\partial f}{\partial x} + ny\frac{\partial f}{\partial y}\right).$$

$$zy'' - z''y = F(x, zy'-z'y), \qquad z(x)\frac{\partial f}{\partial y}.$$

$$\frac{y''}{\Phi} - \frac{y'^2\Phi'}{\Phi^2} = F\left(x, \frac{y'}{\Phi}\right), \qquad \Phi(y)\frac{\partial f}{\partial y}.$$

$$\frac{y''xy^2}{y'^3} + \frac{(n-1)xy}{y'} = F\left(\frac{x}{y^n}, \frac{y}{y'} - nx\right), \qquad y^n\left(nx\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right).$$

$$\frac{y''}{y'^3} + \frac{z''}{z}x = F\left(y, \frac{1}{y'} - \frac{z'}{z}x\right), \qquad z(y)\frac{\partial f}{\partial x}.$$

$$y''\Phi^2 + y'\Phi\Phi' = F(y, y'\Phi), \qquad \Phi(x)\frac{\partial f}{\partial x},$$

the last three being obtained from the preceding three by interchanging x and y and hence replacing y' by $1/y'$, y'' by $-y''/y'^3$.

46. **Second method of integration.** Let the equation be

$$y'' - \omega(x, y, y') = 0. \tag{10}$$

in solved form. Since

$$\frac{dx}{1} = \frac{dy}{y'} = \frac{dy'}{\omega(x, y, y')},$$

the corresponding partial differential equation is

$$Pf = \frac{\partial f}{\partial x} + y'\frac{\partial f}{\partial y} + \omega(x, y, y')\frac{\partial f}{\partial y'} = 0. \tag{11}$$

The first and second extensions of Uf are

$$U'f \equiv \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}, \qquad U''f \equiv U'f + \eta'' \frac{\partial f}{\partial y''}.$$

By the theorem of § 43, $F \equiv y'' - \omega = 0$ is invariant under Uf if and only if $U''F \equiv \eta'' - U'\omega$ vanishes identically in x, y, y' after y'' has been replaced by ω. Inserting the value (4) of η'', we see that the condition becomes

$$\frac{\partial \eta'}{\partial x} + \frac{\partial \eta'}{\partial y} y' + \left(\frac{\partial \eta'}{\partial y'} - \frac{d\xi}{dx} \right) \omega - U'\omega \equiv 0. \tag{12}$$

The sum of the first three terms is $P\eta'$. Hence (10) is invariant under Uf if and only if

$$P\eta' - \omega \frac{d\xi}{dx} - U'\omega \equiv 0, \tag{13}$$

identically in x, y, y'. The commutator of $U'f$ with Pf is

$$-\left(\frac{\partial \xi}{\partial x} + y' \frac{\partial \xi}{\partial y} \right) \frac{\partial f}{\partial x} + \left(\eta' - \frac{\partial \eta}{\partial x} - y' \frac{\partial \eta}{\partial y} \right) \frac{\partial f}{\partial y} + (U'\omega - P\eta') \frac{\partial f}{\partial y'}.$$

Replacing η' by its value (3), we get

$$(U'P)f \equiv -\frac{d\xi}{dx} \frac{\partial f}{\partial x} - y' \frac{d\xi}{dx} \frac{\partial f}{\partial y} + (U'\omega - P\eta') \frac{\partial f}{\partial y'} \equiv -\frac{d\xi}{dx} Pf,$$

if and only if (13) holds. Hence by Theorem 7 of § 37 we deduce the

THEOREM. *The differential equation $y'' = \omega(x, y, y')$ is invariant under the infinitesimal transformation Uf if and only if the corresponding partial differential equation* (11) *is invariant under the first extension $U'f$ of Uf. A convenient form of this condition is* (12).

EXAMPLE. The system of conics $ax^2 + by^2 = 1$ is evidently invariant under the transformations $x_1 = rx$, $y_1 = sy$. Take s as a function of r and differentiate the two equations with respect to r. Write n for the value of ds/dr when $r = 1$. We obtain the infinitesimal transformation

$$Uf \equiv x \frac{\partial f}{\partial x} + ny \frac{\partial f}{\partial y}, \qquad U'f \equiv x \frac{\partial f}{\partial x} + ny \frac{\partial f}{\partial y} + (n-1)y' \frac{\partial f}{\partial y'}. \tag{14}$$

To obtain the differential equation of the conics, differentiate $ax^2+by^2=1$ twice and eliminate a and b. We get

$$\begin{vmatrix} x^2 & y^2 & -1 \\ x & yy' & 0 \\ 1 & y'^2+yy'' & 0 \end{vmatrix} = 0,$$

or

$$y'' = \frac{y'}{x} - \frac{y'^2}{y}. \tag{15}$$

The corresponding partial differential equation (11) is

$$Pf = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + \left(\frac{y'}{x} - \frac{y'^2}{y} \right) \frac{\partial f}{\partial y'} = 0.$$

To check that it is invariant under $U'f$, note that

$$(U'P)f = -Pf.$$

Thus $Pf=0$ and $U'f=0$ form a complete system and this fact simplifies the solution of $Pf=0$. The simplest $U'f$ is that with $n=1$:

$$U'f = Uf = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}.$$

Evident solutions of $U'f=0$ are $u=y/x$, $v=y'$. Some function $F(u,v)$ of these must be a solution also of $Pf=0$. Thus

$$0 = PF = \frac{\partial F}{\partial u} Pu + \frac{\partial F}{\partial v} Pv = \frac{\partial F}{\partial u} \left(\frac{y'}{x} - \frac{y}{x^2} \right) + \frac{\partial F}{\partial v} \left(\frac{y'}{x} - \frac{y'^2}{y} \right).$$

Removing the common factor $y'-yx$, and multiplying by y, we get

$$u \frac{\partial F}{\partial u} - v \frac{\partial F}{\partial v} = 0,$$

which has the evident solution $uv=yy'/x$.

To find another solution of $Pf=0$, we employ the variables $x, y, z=yy'/x$. Then $Pf=0$ and $U'f$ become

$$\frac{\partial f}{\partial x} + \frac{xz}{y} \frac{\partial f}{\partial y} = 0, \qquad x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}.$$

Since the former is invariant under the latter, $xz\,dx-y\,dy=0$ has the integrating factor $1/(x^2z-y^2)$ by § 15, whence x^2z-y^2 is a solution. From the resulting two first integrals

$$\frac{yy'}{x} = c, \qquad xyy'-y^2 = d$$

of (15), we eliminate y' and obtain the general integral $cx^2-y^2=d$ of (15), and hence the initial conics with altered parameters.

47. **Linearly independent or dependent infinitesimal transformations.** We shall call $U_1 f, \ldots, U_s f$ linearly dependent if there exist constants $k_1, \ldots, k_s$ not all zero such that $k_1 U_1 f + \cdots + k_s U_s f = 0$, but linearly independent if no such constants k_i exist. In the first case one of the $U_i f$ can be expressed linearly in terms of the remaining U's; if the latter are linearly dependent we repeat the process. Hence we may assume that $U_1 f, \ldots, U_k f$, for example, are linearly independent, while $U_{k-1} f, \ldots, U_s f$ are expressible linearly in terms of the former.

LEMMA. *If $y'' = \omega(x, y, y')$ is invariant under $U_1 f, \ldots, U_k f$, it is invariant under every transformation*

$$(16) \qquad U f = c_1 U_1 f + \cdots + c_k U_k f \qquad (c_1, \ldots, c_k \text{ constants}),$$

which is linearly dependent on them.

For, by § 46, their first extensions $U_1' f, \ldots, U_k' f$ leave (11) invariant, so that

$$(17) \qquad (U_1' P) f = \varrho_1 P f, \ldots, \qquad (U_k' P) f = \varrho_k P f.$$

We see at once from (3) that the first extension of (16) is

$$U' f = c_1 U_1' f + \cdots + c_k U_k' f.$$

By (17),

$$(U' P) f = c_1 (U_1' P) f + \cdots + c_k (U_k' P) f = (c_1 \varrho_1 + \cdots + c_k \varrho_k) P f.$$

Hence by Theorem 7 of § 37, $U' f$ leaves $P f = 0$ invariant. By § 46, $U f$ therefore leaves $y'' = w$ invariant.

Hence (16) may be discarded since it furnishes no information about $y'' = w$ which is not already implied by its invariance under $U_1 f, \ldots, U_k f$.

48. **Maximum number of linearly independent infinitesimal transformations Uf leaving $y'' = \omega$ invariant.** We shall assume that the given function $\omega(x, y, y')$ may be expanded into a series involving only positive powers of y'. If we insert the expression (3) for η' into (12), we get

$$(18) \qquad \begin{aligned} &\eta_{xx} + (2\eta_{xy} - \xi_{xx}) y' + (\eta_{yy} - 2\xi_{xy}) y'^2 - \xi_{yy} y'^3 - \xi \omega_x - \eta \omega_y \\ &\quad + (\eta_y - 2\xi_x - 3\xi_y y') \omega - [\eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2] \omega_{y'} = 0, \end{aligned}$$

in which subscripts denote partial differentiation. Replace ω by its expansion in a series. Then the total coefficient of each power of y' in our identity must be zero. By the terms free of y' and the coefficients of y', y'^2, y'^3, we evidently get

$$(19) \qquad \eta_{xx} = f_0, \qquad 2\eta_{xy} - \xi_{xx} = f_1, \qquad \eta_{yy} - 2\xi_{xy} = f_2, \qquad \xi_{yy} = f_3,$$

where each f_i is a linear homogeneous function of

$$\xi, \quad \eta, \quad \xi_x, \quad \xi_y, \quad \eta_x, \quad \eta_y, \tag{20}$$

whose coefficients are known functions of x and y. By the coefficients of $y'^4, y'^5, \ldots$, we obtain linear relations involving the functions (20) which together with (19) are necessary and sufficient conditions on ξ and η in order that

$$Uf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}$$

shall leave $y'' = \omega$ invariant.

The eight relations obtained by differentiating each of the four equations (19) partially with respect to x and y in turn are seen at once to give the values of the eight partial derivatives $\eta_{xxx}, \eta_{xxy}, \ldots, \xi_{yyy}$ of the third order expressed as linear homogeneous functions of (20) and the six second partial derivatives of ξ and η.

If we assign the values at $x = 0, y = 0$ of the functions (20) and ξ_{xx} and ξ_{xy}, we may compute by (19) and their successive derivatives the values of all second and higher derivatives of ξ and η, and hence determine ξ and η by means of

$$f(x, y) = f(0, 0) + x f_x + y f_y + \tfrac{1}{2} x^2 f_{xx} + xy f_{xy} + \tfrac{1}{2} y^2 f_{yy} + \cdots,$$

where f_x denotes the value of f_x at $x = 0, y = 0$, etc.

THEOREM.* *A differential equation of the second order is invariant under at most 8 linearly independent infinitesimal transformations.*

In the above discussion we attended only to the conditions (19) and ignored the further linear relations between the functions (20). The number of arbitrary constants in the general solution ξ, η of all these relations may be fewer than 8. Moreover, these relations may not be consistent. In fact, let

$$\omega(x, y, y') = g(x, y) + e^{y'}.$$

Then the coefficient of $e^{y'}$ in (18) is

$$(\eta_y - 2\xi_x - 3\xi_y y') - [\eta_x + (\eta_y - \xi_x) y' - \xi_y y'^2] = 0,$$

whence

$$\xi_y = 0, \qquad \eta_y = \xi_x = -\eta_x.$$

* For a geometrical proof, see Lie-Scheffers, Continuierliche Gruppen, pp. 294–8. It is proved also that an ordinary differential equation of order r ($r > 2$) is invariant under at most $r+4$ linearly independent infinitesimal transformations. This maximum is reached for $y^{(r)} = 0$.

Thus $\xi_{xx} = 0$, $\xi = ax + b$, $\eta = a(y - x) + c$, where a, b, c, are constants. Thus (18) becomes $-\xi g_x - \eta g_y - \xi_x g = 0$. Take $g = xy$. Then $a = b = c = 0$, whereas ξ and η are not both zero identically. Hence $y'' = xy + e^y$ *is invariant under no infinitesimal transformation.*

But for $\omega = 0$, (18) holds if and only if conditions (19) hold with each $f_i = 0$. All eight partial derivatives of the third order of ξ and η are now identically zero. Hence ξ and η are polynomials of the second degree in x and y. Then (19) with each $f_i = 0$ are satisfied if and only if

$$(21)\quad \xi = C_1 + D_1 x + B_1 y - A_3 x^2 - B_3 xy, \quad \eta = C_2 + A_2 x + D_2 y - A_3 xy - B_3 y^2.$$

Hence* $y'' = 0$ *is invariant under exactly* 8 *linearly independent infinitesimal transformations.*

49. **Differential equation of the second order invariant under two infinitesimal transformations.** We shall begin with illustrative examples whose discussion discloses the advantages to be gained from a knowledge of two infinitesimal transformations each leaving invariant the differential equation.

Example 1. We shall treat equation (15) from a new standpoint. From the start, we knew that it is invariant under the infinitude of infinitesimal transformations (14), or, what is equivalent, under the two linearly independent transformations

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \qquad y\frac{\partial f}{\partial y}.$$

In other words, $Pf = 0$ is invariant under their first extensions

$$Uf = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}, \qquad Vf = y\frac{\partial f}{\partial y} + y'\frac{\partial f}{\partial y'}.$$

* This follows also from the fact that the integral curves of $y'' = 0$ are all the straight lines in the plane. These lines are merely permuted by the so-called projective transformations

$$x_1 = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \qquad y_1 = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}.$$

We obtain a one-parameter group by taking $a_1, \ldots, c_3$ to be functions of t such that $a_1 = b_2 = c_3 = 1$, $b_1 = c_1 = a_2 = c_2 = a_3 = b_3 = 0$ when $t = 0$ (whence $x_1 = x$, $y_1 = y$). Then

$$\left(\frac{dx_1}{dt}\right)_{t=0} = A_1 x + B_1 y + C_1 - x(A_3 x + B_3 y + C_3),$$

$$\left(\frac{dy_1}{dt}\right)_{t=0} = A_2 x + B_2 y + C_2 - y(A_3 x + B_3 y + C_3),$$

if A_1, B_1, ... denote the values of the derivatives of a_1, b_1, ... when $t = 0$. Writing D_1 for $A_1 - C_3$, and D_2 for $B_2 - C_3$, we obtain (21).

Hence we may apply the method of case (ii) in § 41 with $(UV)f = 0$. We shall omit the details of the integrations since full details will be given in the next two examples, which are more interesting than this one. We find that

$$\chi = \log \frac{yy'}{x}, \qquad \psi = \log \frac{y'}{x(y - xy')},$$

whose difference is $\log(y^2 - xyy')$. We therefore obtain the same results as in our earlier treatment of this example.

EXAMPLE 2. To integrate $y'' = X(x)y' + Q(x)y + R(x)$.

Let $z(x)$ and $w(x)$ be two linearly independent particular solutions of the abridged equation $y'' = Xy' + Qy$ (see Ex. 3). Then

$$Pf = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y} + (Xy' + Qy + R) \frac{\partial f}{\partial y'} = 0$$

is invariant under the following two linearly independent infinitesimal transformations (§ 45, Ex. 2):

$$Uf = z \frac{\partial f}{\partial y} + z' \frac{\partial f}{\partial y'}, \qquad Vf = w \frac{\partial f}{\partial y} + w' \frac{\partial f}{\partial y'}.$$

The problem falls under case (ii) in § 41 since

$$(UV)f = 0, \quad \Delta = \begin{vmatrix} 1 & y' & Xy' + Qy + R \\ 0 & z & z' \\ 0 & w & w' \end{vmatrix} = zw' - z'w \not\equiv 0.$$

For, if $\Delta = 0$, then $dw : w = dz : z$, $w = cz$, where c is a constant, contrary to the assumed linear independence of z and w. Next,

$$z'' = Xz' + Qz, \qquad w'' = Xw' + Qw$$

give

$$X = \frac{zw'' - wz''}{\Delta}, \qquad Q = \frac{w'z'' - z'w''}{\Delta} = \frac{z''\Delta - z'\Delta'}{z\Delta}.$$

Inserting these values in the formula for the common solution

$$\chi = \int \frac{\begin{vmatrix} dx & dy & dy' \\ 1 & y' & Xy' + Qy + R \\ 0 & z & z' \end{vmatrix}}{\Delta} \tag{22}$$

of $Pf = 0$ and $Uf = 0$, we get

$$\chi = \int \frac{[(z'y' - z''y)\Delta - (zy' - z'y)\Delta'](dx - z'\Delta\, dy + z\Delta\, dy')}{\Delta^2} - \int \frac{zR\,dx}{\Delta}.$$

Since the first integrand is an exact differential, we get

$$\chi = \frac{zy' - z'y}{\Delta} - \int \frac{zR\,dx}{\Delta}.$$

By interchanging z with w (and z' with w'), we deduce the second solution

$$\psi = \frac{w y' - w' y}{-\Delta} - \int \frac{w R \, dx}{-\Delta}.$$

Eliminating y' between the first integrals $\chi = a$ and $\psi = b$, we get the following general integral of the proposed differential equation:

$$y = a w + b z + w \int \frac{z R \, dx}{\Delta} - z \int \frac{w R \, dx}{\Delta}.$$

EXAMPLE 3. Given a particular solution $z(x)$, to integrate

$$y'' = N(x) y' + Q(x) y.$$

As in Ex. 2, the corresponding partial differential equation $Pf = 0$ is invariant under Uf. By Ex. 1 of § 45, it is invariant under also

$$Vf = y \frac{\partial f}{\partial y} + y' \frac{\partial f}{\partial y'}.$$

Here $\Delta = z y' - z' y$ is not identically zero in the independent variables x, y, y', since $z \not\equiv 0$. Also $(UV)f = Uf$. Hence the problem falls under case (ii) of § 41. We have (22) with $R = 0$. Replacing Qz by its value $z'' - Nz'$, we find that the integrand in (22) is the total derivative of χ:

$$\chi = \log \Delta - \int N \, dx, \qquad r = e^{\chi} = \Delta e^{-\int N \, dx}.$$

Introduce the variables x, y, r. Then $Pf = 0$ becomes

$$\frac{\partial f}{dx} + \frac{1}{z} \left(z' y + r e^{\int N \, dx} \right) \frac{\partial f}{\partial y} = 0.$$

When r is regarded as a constant, $Pf = 0$ is invariant under

$$z \frac{\partial f}{\partial y},$$

since the commutator is identically zero. Hence $1/z$ is an integrating factor of the corresponding ordinary differential equation

$$\frac{1}{z} \left(z' y + r e^{\int N \, dx} \right) dx - dy = 0,$$

whose integral is therefore

$$\frac{y}{z} - r \int z^{-2} e^{\int N \, dx} \, dx = c.$$

This, with r taken as a constant, is the general integral.

In each of these three examples, the differential equation of the second order is invariant under two infinitesimal transformations Uf and Vf such that

$$(UV)f = aUf + bVf, \tag{23}$$

where a and b are constants, one or both of which may be zero.

Our next aim is to prove that, when the two given infinitesimal transformations leaving invariant a differential equation of the second order do not have the property (23), we can readily determine from them two new infinitesimal transformations which have that property and leave the equation invariant. We shall then be in a position to specify the four subcases necessary in an exhaustive treatment of differential equations of the second order invariant under two given infinitesimal transformations.

50. **First extension of the commutator** $(UV)f$. We have

$$U'f = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}, \qquad \eta' = \frac{d\eta}{dx} - y' \frac{d\xi}{dx}.$$

If g is an arbitrary function of x and y, write

$$Bg = \frac{dg}{dx} = \frac{\partial g}{\partial x} + y' \frac{\partial g}{\partial y}.$$

Hence $\eta' = B\eta - y' B\xi$. We have

$$(BU')f = B\xi \frac{\partial f}{\partial x} + y' B\xi \frac{\partial f}{\partial y} + \varrho \frac{\partial f}{\partial y'},$$

where the expression for ϱ will not be needed. Hence

$$(BU')f = B\xi \cdot Bf + \varrho Cf, \qquad Cf = \frac{\partial f}{\partial y'}.$$

Evidently $(CU')f = \sigma Cf$, where the expression for σ will not be needed.

Similarly, $(BV')f$ and $(CV')f$ are linear combinations of Bf and Cf. By Jacobi's identity in § 39,

$$((U'V')B)f + ((V'B)U')f + ((BU')V')f = 0.$$

The third term of this is equal to

$$(\alpha B + \varrho C, V')f = \alpha(BV')f - V'\alpha \cdot Bf + \varrho(CV')f - V'\varrho \cdot Cf$$

by (31) of § 38, and hence is a linear combination of Bf and Cf. The same is true of the second term of Jacobi's identity, which therefore gives

$$(24) \qquad (B(U'V'))f = \lambda Bf + \mu Cf = \lambda \frac{\partial f}{\partial x} + \lambda y' \frac{\partial f}{\partial y} + \mu \frac{\partial f}{\partial y'}.$$

Let

$$(UV)f = \xi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y}.$$

In view of the foregoing expression for η', we have

$$(25) \qquad (UV)'f = \xi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y} + (B\eta_1 - y' B\xi_1) \frac{\partial f}{\partial y'}.$$

Since the two terms of the commutator $(UV)f$ are (§ 21) the first two of the three terms of the commutator of $U'f$ with $V'f$, we have

$$(26) \qquad (U'V')f = \xi_1 \frac{\partial f}{\partial x} + \eta_1 \frac{\partial f}{\partial y} + \zeta \frac{\partial f}{\partial y'},$$

in which ζ may be found indirectly as follows. We find that

$$(B(U'V'))f = B\xi_1 \frac{\partial f}{\partial x} + (B\eta_1 - \zeta) \frac{\partial f}{\partial y} + \tau \frac{\partial f}{\partial y'},$$

in which the expression for τ will not be needed. Comparing this with (24), we see that

$$\lambda = B\xi_1, \qquad \lambda y' = B\eta_1 - \zeta, \qquad \zeta = B\eta_1 - y' B\xi_1,$$

so that (25) and (26) are identical.

THEOREM. *The commutator $(U'V')f$ of the first extensions of two infinitesimal transformations Uf and Vf is identical with the first extension $(UV)'f$ of their commutator $(UV)f$.*

Corollary. If $y'' = \omega(x, y, y')$ is invariant under both Uf and Vf it is invariant also under their commutator $(UV)f$.

For, the corresponding partial differential equation $Pf = 0$, given by (11), is then invariant under their first extensions $U'f$ and $V'f$ and hence (§ 40) under $(U'V')f$. Since $Pf = 0$ is therefore invariant under $(UV)'f$, $y'' = \omega$ is invariant under $(UV)f$ by § 46.

51. **Closed system of infinitesimal transformations leaving $y'' = \omega$ invariant.** Given several infinitesimal transformations leaving $y'' = \omega$ invariant, we may discard (§ 47) those which are linearly dependent on the remaining. Hence it suffices to consider linearly independent infinitesimal transformations $U_1 f, \ldots, U_k f$ leaving $y'' = \omega$ invariant.

By the preceding corollary, it is invariant under $(U_1 U_2)f$, which we discard if it is linearly dependent on $U_1 f, \ldots, U_k f$. But in the contrary case we annex it to them and obtain a larger set $U_1 f, \ldots, U_{k+1} f$ of linearly independent infinitesimal transformations leaving $y'' = \omega$ invariant. If the commutator of any two of the latter set is linearly independent of these $k+1$ transformations, we annex it and obtain a still larger set. Proceeding in this manner, we ultimately reach a finite number (§ 48) of

linearly independent infinitesimal transformations $U_1 f, \ldots, U_r f$ leaving $y'' = \omega$ invariant and such that the commutator of any two of them is a linear combination of them with constant coefficients. Then all their linear combinations $c_1 U_1 f + \ldots + c_r U_r f$ with constant coefficients $c_1, \ldots, c_r$ are said to form a *closed system** of infinitesimal transformations.

THEOREM. *If two or more infinitesimal transformations leave invariant a differential equation of the second order, they are contained in a closed system leaving it invariant. In particular, the totality of infinitesimal transformations leaving it invariant forms a closed system.*

For example, $y'' = 0$ is invariant under

$$Uf = x \frac{\partial f}{\partial y}, \qquad Vf = y \frac{\partial f}{\partial x},$$

since

$$U'f = x \frac{\partial f}{\partial y} + \frac{\partial f}{\partial y'}, \qquad V'f = y \frac{\partial f}{\partial x} - y'^2 \frac{\partial f}{\partial y'}, \qquad Pf = \frac{\partial f}{\partial x} + y' \frac{\partial f}{\partial y},$$

$$(U'P)f = 0, \qquad (V'P)f = -y' Pf.$$

By the corollary in § 50, $y'' = 0$ is invariant also under Wf, where

$$Wf = (UV)f = x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y}, \qquad W'f = x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} - 2y' \frac{\partial f}{\partial y'}.$$

This follows also from $(W'P)f = -Pf$. Here Uf, Vf, Wf are evidently linearly independent; their linear combinations with constant coefficients form a closed system since

$$(UV)f = Wf, \qquad (UW)f = -2Uf, \qquad (VW)f = 2Vf.$$

Note also the five closed systems formed by the linear combinations of

$$Uf, Wf; \quad Vf, Wf; \quad U'f, Pf; \quad W'f, Pf; \quad U'f, W'f, Pf.$$

All infinitesimal transformations which leave $y'' = 0$ invariant are determined by (21). Hence they form a closed system. It is not very laborious to verify this fact by showing that the commutator of any two of them is one of them.

52. Closed systems determined by two infinitesimal transformations. We are now in a position to prove the following statement made at the end of § 49:

THEOREM 1. *If we know two infinitesimal transformations Uf and Wf which leave invariant a given differential equation $y'' = \omega$, we can find another infinitesimal transformation Vf leaving it invariant such that the linear combinations of Uf and Vf with constant coefficients form a closed system, viz.,*

$$(UV)f = aUf + bVf \qquad (a, b \text{ constants}). \tag{27}$$

* Often called an r-parameter group of infinitesimal transformations. But it is neither a group (in the technical sense), nor does it have r essential parameters, since the $r-1$ ratios $c_1 : c_2 : \ldots : c_r$ give the only essential parameters. But the closed system generates an r-parameter group composed of ∞^r transformations only ∞^{r-1} of which are infinitesimal and constitute our closed system.

By the proof of the Theorem in § 51, we can find r linearly independent infinitesimal transformations $U_1 f = Uf, U_2 f, \ldots, U_r f$ leaving $y'' = \omega$ invariant such that

$$(U_i U_j) f = \sum_{k=1}^{r} c_{ijk} U_k f \qquad (i, j = 1, \ldots, r),$$

where the c_{ijk} are constants. We seek constants $a, b, e_2, \ldots, e_r$ such that (27) shall hold for $U = U_1$ and

$$Vf = e_2 U_2 f + \cdots + e_r U_r f.$$

We have

$$(UV) f = \sum_{j=2}^{r} e_j (U_1 U_j) f = \sum_{j=2}^{r} e_j \sum_{k=1}^{r} c_{1jk} U_k f.$$

We desire that this shall be identical with the right member of (27), viz.,

$$a U_1 f + b \sum_{k=2}^{r} e_k U_k f.$$

In view of the linear independence of $U_1 f, \ldots, U_r f$, the conditions are

$$\sum_{j=2}^{r} e_j c_{1j1} = a, \quad \sum_{j=2}^{r} e_j c_{1jk} = b e_k \qquad (k = 2, \ldots, r). \tag{28}$$

The first condition will serve to determine a after the e_j are found. The remaining $r-1$ conditions (28) may be written in the form

$$\begin{aligned}
(c_{122} - b) e_2 + c_{132} e_3 + \cdots + c_{1r2} e_r &= 0, \\
c_{123} e_2 + (c_{133} - b) e_3 + \cdots + c_{1r3} e_r &= 0, \\
\cdots\cdots\cdots

The expanded form of this relation is an algebraic equation of degree $r-1$ in b in which the coefficient of b^{r-1} is $(-1)^{r-1}$. Hence whatever be the values of the given c_{ijk}, we have an equation actually involving b and therefore having at least one root b. This proves Theorem 1.

The proof leads also to a more abstract result:

THEOREM 2. *Any infinitesimal transformation Uf of any closed system lies in a closed system formed by the linear combinations of Uf and another transformation of the first system.*

For examples illustrating this theorem see the end of § 51.

We shall now obtain canonical forms for closed systems determined by two infinitesimal transformations Rf, Sf on x, y. By hypothesis, we have $(RS)f = \lambda Rf + \mu Sf$, where λ and μ are constants. If $\lambda = \mu = 0$, take $U = R$, $V = S$, whence $(UV)f = 0$. If $\lambda \neq 0$, $\mu = 0$, take $U = R$, $V = \lambda^{-1} S$, whence $(UV)f = Uf$. If $\mu \neq 0$, take

$$Uf = \frac{\lambda}{\mu} Rf + Sf, \quad Vf = -\frac{1}{\mu} Rf,$$

whence $(UV)f = Uf$. Hence in every case

$$(UV)f = kUf, \qquad k = 0 \text{ or } 1. \tag{29}$$

By § 22, this implies that the partial differential equation $Uf = 0$ is invariant under the infinitesimal transformation Vf. Hence a solution $u(x, y)$ of $Uf = 0$ can be found by quadratures (§ 15). After interchanging x and y if necessary, we may assume that x actually occurs in $u(x, y)$. In terms of the new variables u and y, Uf becomes

$$U_1 f = Uu \cdot \frac{\partial f}{\partial u} + Uy \cdot \frac{\partial f}{\partial y} = \Phi \frac{\partial f}{\partial y},$$

where Φ is the function obtained by expressing Uy in terms of u and y. Finally, we introduce the new variables u and

$$Y = \int^{\cdot} \frac{dy}{\Phi}.$$

in the integration of which u plays the rôle of a constant. Since $U_1 Y = 1$, $U_1 f$ becomes $\partial f / \partial Y$. Hence we can determine new variables by quadratures such that Uf becomes*

$$Uf = \frac{\partial f}{\partial y}. \tag{30}$$

* This choice (30) instead of $\partial f / \partial x$ simplifies the work in § 53.

Then, by (29),

$$Vf = \xi \frac{\partial f}{\partial x} + \eta \frac{\partial f}{\partial y}, \qquad \frac{\partial \xi}{\partial y} = 0, \qquad \frac{\partial \eta}{\partial y} = k,$$

whence

$$Vf = \sigma(x) \frac{\partial f}{\partial x} + \{ky + \varrho(x)\} \frac{\partial f}{\partial y}.$$

First, let $\sigma(x) = 0$. If $k = 0$, then $\varrho(x)$ is not identically zero, since Vf is not identically zero. Replacing $\varrho(x)$ by a new variable x, we have

$$\text{(I)} \qquad Uf = \frac{\partial f}{\partial y}, \qquad Vf = x \frac{\partial f}{\partial y}, \qquad (UV)f = 0.$$

But if $k = 1$, we replace $y + \varrho(x)$ by a new variable y and get

$$\text{(II)} \qquad Uf = \frac{\partial f}{\partial y}, \qquad Vf = y \frac{\partial f}{\partial y}, \qquad (UV)f = Uf.$$

Second, let $\sigma(x)$ be not identically zero. Introduce the variables

$$x_1 = \psi(x), \qquad y_1 = y - \Phi(x),$$

where $\psi(x)$ is not a constant. Then Uf and Vf become

$$U_1 f = \frac{\partial f}{\partial y_1},$$

$$V_1 f = \sigma(x)\psi'(x) \frac{\partial f}{\partial x_1} + \{ky_1 + k\Phi(x) + \varrho(x) - \sigma(x)\Phi'(x)\} \frac{\partial f}{\partial y_1},$$

in which theoretically x should be replaced by its value in terms of x_1 obtained by solving $x_1 = \psi(x)$, but practically this replacement is not necessary here. If $k = 0$, we take

$$\psi(x) = \int^x \frac{dx}{\sigma(x)}, \qquad \Phi(x) = \int^x \frac{\varrho(x)}{\sigma(x)} dx,$$

and, after dropping the subscripts 1, obtain

$$\text{(III)} \qquad Uf = \frac{\partial f}{\partial y}, \qquad Vf = \frac{\partial f}{\partial x}, \qquad (UV)f = 0.$$

But if $k = 1$, we can choose Φ and ψ so that

$$V_1 f = x_1 \frac{\partial f}{\partial x_1} + y_1 \frac{\partial f}{\partial y_1},$$

the conditions for which are

$$\sigma(x)\psi'(x) = \psi(x), \qquad \Phi(x) + \varrho(x) - \sigma(x)\Phi'(x) = 0.$$

The latter is a linear differential equation for Φ; by § 16, we get

$$\psi(x) = e^{\int \frac{dx}{\sigma(x)}}, \qquad \Phi(x) = e^{\int \frac{dx}{\sigma(x)}} \int \frac{\varrho(x)}{\sigma(x)} e^{-\int \frac{dx}{\sigma(x)}} dx.$$

Dropping the subscripts 1, we have

$$\text{(IV)} \qquad Uf = \frac{\partial f}{\partial y}, \qquad Vf = x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y}, \qquad (UV)f = Uf.$$

THEOREM 3. *Given two infinitesimal transformations Uf and Vf on x and y for which $(UV)f = kUf$, $k = 0$ or 1, we can find by quadratures* alone new variables such that the infinitesimal transformations take one of the four canonical forms* (I)–(IV), *which are distinguished from one another by the value of k and the fact that Vf is the product of Uf by a function of x and y in cases* (I) *and* (II), *but not in cases* (III) *and* (IV).

Instead of first reducing Uf to the canonical form (30) and then reducing Vf to one of its four canonical forms in (I)–(IV) without altering (30), we may perform the reductions simultaneously and in fact often more simply. For example, let

$$(UV)f = 0, \qquad Vf = \rho Uf.$$

Our proof shows that there exist new variables x_1, y_1 for which

$$\text{(I)} \qquad Uf = \frac{\partial f}{\partial y_1}, \qquad Vf = x_1 \frac{\partial f}{\partial y_1} = x_1 Uf.$$

Hence $x_1 = \rho$ is known. Since $0 = (U, \rho U)f = U\rho \cdot Uf$, we have $U\rho = 0$. In terms of the new variables $x_1 = \rho$ and y_1, Uf therefore becomes

$$Uy_1 \cdot \frac{\partial f}{\partial y_1}.$$

* In Lie-Scheffers, Differentialgleichungen, pp. 424–425, it is stated incorrectly that the reduction to (IV) is not accomplished by quadratures alone, but that we must solve a differential equation (equivalent to $Uf = 0$). This oversight was caused by the ignoring of one of the hypotheses, viz., $(UV) = Uf$.

Hence we desire a solution y_1 of

$$Uy_1 = \xi \frac{\partial y_1}{\partial x} + \eta \frac{\partial y_1}{\partial y} = 1,$$

which is equivalent to the simultaneous system*

$$\frac{dx}{\xi} = \frac{dy}{\eta} = dy_1.$$

But the first of these equations is equivalent to $Uf = 0$, which has the known solution ρ. Eliminating x, for example, by means of $\rho = c$, taken in the solved form $x = \Phi(y, c)$, we get

$$\frac{dy}{\eta(\Phi, y)} = dy_1, \qquad y_1 = \int \frac{dy}{\eta(\Phi, y)}.$$

53. **Integration of a differential equation of the second order invariant under two infinitesimal transformations.** In view of Theorem 1 of § 52, we may assume that the two transformations define a closed system, and then, by (29), select transformations Uf and Vf from the system such that $(UV)f = kUf$, $k = 0$ or 1. By Theorem 3 of § 52, new variables may be found by quadratures such that Uf and Vf take one of the four canonical forms (I)–(IV). Let the differential equation become $y'' = \omega(x, y, y')$ when written in our new variables. We apply the condition (12) that it be invariant under our $Uf = \partial f / \partial y$, for which $\xi = \eta' = 0$, and get $\partial \omega / \partial y = 0$. Henceforth we shall suppress this derivative without further notice.

For Vf in (I), we have $\xi = 0$, $\eta = x$, $\eta' = 1$. Then (12) becomes $V'\omega = 0$, or $\partial \omega / \partial y' = 0$. Thus

(i) $$y'' = \omega(x), \qquad y = \int \left\{ \int \omega(x)\, dx \right\} dx + ax + b.$$

For Vf in (II), $\xi = 0$, $\eta = y$, $\eta' = y'$, and (12) becomes $\omega - V'\omega = 0$, or

$$\omega - y' \frac{\partial \omega}{\partial y'} = 0, \qquad \frac{\partial \log \omega}{\partial y'} = \frac{1}{y'} = \frac{\partial \log y'}{\partial y'}, \qquad \frac{\omega}{y'} = f(x),$$

(ii) $$y'' = y' f(x), \quad \log y' = \int f(x)\, dx + \log a, \quad y = a \int e^{\int f(x)\, dx}\, dx + b.$$

* Since $dy_1 = \frac{\partial y_1}{\partial x} dx + \frac{\partial y_1}{\partial y} dy = Uy_1 \cdot \frac{dx}{\xi}$.

For Vf in (III), $\xi = 1$, $\eta = 0$, $\eta' = 0$, and (12) becomes $V'\omega = 0$ or $\partial\omega/\partial x = 0$.

$$\text{(iii)}\qquad \begin{aligned} y'' &= \omega(y'), & \int \frac{dy'}{\omega(y')} &= x + a, \\ y' &= \Phi(x+a), & y &= \int \Phi(x+a)\,dx + b. \end{aligned}$$

For Vf in (IV), $\xi = x$, $\eta = y$, $\eta' = 0$, and (12) becomes $-\omega - V'\omega = 0$, or

$$\omega + x\frac{\partial\omega}{\partial x} = 0, \qquad \frac{\partial(\omega x)}{\partial x} = 0, \qquad \omega x = f(y'),$$

$$\text{(iv)}\qquad \begin{aligned} y'' &= \frac{1}{x} f(y'), & \int \frac{dy'}{f(y')} &= \log x + a, \\ y' &= \Phi(\log x + a), & y &= \int \Phi(\log x + a)\,dx + b. \end{aligned}$$

THEOREM. *If a differential equation of the second order is invariant under two infinitesimal transformations, it can be integrated by quadratures.**

For example, consider the linear differential equation

$$y'' = X(x)\,y' + Q(x)\,y + R(x).$$

By Ex. 2 of § 49, it is invariant under

$$Uf = z\frac{\partial f}{\partial y}, \qquad Vf = w\frac{\partial f}{\partial y}, \qquad (UV)f = 0,$$

where z and w are two linearly independent particular solutions of the abridged equation $y'' = Xy' + Qy$. The canonical transformations are therefore (I) written in new variables x_1, y_1. Evidently $x_1 = w/z$. For $y_1 = y/z$, Uf and Vf take their canonical forms (I). It remains to express the differential equation in the new variables. We find that

$$y_1' = \frac{dy_1}{dx_1} = \frac{zy' - yz'}{\Delta}, \qquad \Delta = zw' - wz', \qquad \frac{dx_1}{dx} = \frac{\Delta}{z^2},$$

$$\frac{dy_1'}{dx} = \frac{\Delta(zy'' - yz'') - (zy' - yz')\Delta'}{\Delta^2} = \frac{z}{\Delta}y'' - \frac{z\Delta'}{\Delta^2}y' - \frac{(z''\Delta - z'\Delta')}{\Delta^2}y$$

$$= \frac{z}{\Delta}y'' - \frac{z}{\Delta}Xy' - \frac{z}{\Delta}Qy = \frac{z}{\Delta}R,$$

* On account of the oversight mentioned in the foot-note to Theorem 3 of § 52, Scheffers stated that in case (IV) the integration by the present method requires the solution of a differential equation of the first order. Later in his book (pp. 457–472) he took the (unnecessary) trouble to develop another method which requires only quadratures.

after substituting the values of N and Q obtained in § 49 by solving

$$z'' = Nz' + Qz, \qquad w'' = Nw' + Qw$$

for N and Q. Hence

$$y_1' \quad \frac{dy_1'}{dx_1} = \frac{dy_1'}{dx} \Big/ \frac{dx_1}{dx} = \frac{zR}{\Delta} \Big/ \frac{\Delta}{z^2}, \qquad y_1'' = \frac{z^3 R}{\Delta^2}.$$

Since the final fraction is a function of x only, it can be expressed as a function $F(x_1)$ of x_1 only by means of the solved form of $x_1 = w/z$. Then

$$y_1 = \int dx_1 \int F(x_1)\, dx_1 + ax_1 + b.$$

Returning to the initial variables, we get

$$y = z \int d\left(\frac{w}{z}\right) \int \frac{z^3 R}{(zw' - wz')^2}\, d\left(\frac{w}{z}\right) + aw + bz.$$

Consider the above abridged equation. It is invariant under

$$Uf \quad z\frac{\partial f}{\partial y}, \qquad Vf \quad y\frac{\partial f}{\partial y}, \qquad (UV)f = Uf,$$

by Ex. 3 of § 49. These are reduced to their canonical forms (II), written in new variables x_1, y_1, if we take $x_1 = x$, $y_1 = yz$. Then

$$y = zy_1, \qquad y' = zy_1' + z'y_1, \qquad y'' = zy_1'' + 2z'y_1' + z''y_1,$$

$$y'' - Ny' - Qy = zy_1'' + (2z' - zN)y_1' + (z'' - Nz' - Qz)y_1,$$

and the coefficient of y_1 is identically zero, since z is a particular solution. Hence

$$y_1'' = f(x)y_1', \qquad f(x) = N - 2z'/z, \qquad y_1' = ae^{\int f dx},$$

$$y = zy_1 = az\int e^{\int f dx}\, dx + bz.$$

Chapter V.

Applications to geometry and algebraic invariants.

We presuppose no acquaintance with the subjects to which we shall apply the theory of infinitesimal transformations and groups generated by them; on the contrary, our discussion furnishes luminous introductions to those subjects.

First, we shall prove that two plane curves are congruent, i. e., can be brought into coincidence by a rigid motion of the plane, only when then have the same (intrinsic) equation between their differential invariants

of the second and third orders. Next, we shall obtain very simply a complete set of functionally independent covariants and invariants of any binary form, and at the same time provide an easy introduction to the complicated algebraic theory of invariants.

54. **Differential invariants and the congruence of plane curves.** We shall first determine all differential invariants of the group G of all rigid motions in the plane. Each such motion is the product of a translation $x_1 = x + a$, $y_1 = y + b$, by a rotation about the origin. Hence the group G is generated by the three infinitesimal transformations

$$(1) \qquad Uf = \frac{\partial f}{\partial x}, \qquad Vf = \frac{\partial f}{\partial y}, \qquad Wf = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y}.$$

Differential invariants of the first or second order were defined in § 25 and § 44 to be functions of x, y, $y' = dy/dx$ or of x, y, y', y'' which are invariant under the first or second extensions of our infinitesimal transformations. Evidently Uf and Vf are identical with their extensions. Hence a differential invariant does not involve x or y. The second extension of Wf was seen, at the end of § 43, to be

$$W''f = -y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} + (1 + y'^2)\frac{\partial f}{\partial y'} + 3y'y''\frac{\partial f}{\partial y''}.$$

Hence every differential invariant of G of the first or second order is a solution, not involving x or y, of $W''f = 0$, and hence is an integral of

$$\frac{dy'}{1 + y'^2} = \frac{dy''}{3y'y''}.$$

To separate the variables multiply by y'. Hence every such differential invariant is a function of

$$(2) \qquad k^2 = \frac{y''^2}{(1 + y'^2)^3},$$

which is known to be the square of the curvature k of a plane curve C defined by $y = F(x)$. Let τ be the angle made with the positive x-axis by the tangent to the curve C at the point (x, y) on it. Under any translation $x_1 = x + a$, $y_1 = y + b$, the curve C becomes a curve whose tangent at (x_1, y_1) makes the same angle τ with the x-axis. Under a rotation

through angle θ about the origin. C becomes a curve whose tangent at the rotated point makes the angle $\iota + \theta$ with the x-axis. Hence $d\iota$ is unaltered by all transformations of the group G. We shall exclude the case in which C reduces to a straight line since ι is then constant along the curve. Hence* $dI\,d\iota$ is a differential invariant if I is one, so that also dI is unaltered by G. Since $\iota = \arctan y'$.

$$(3) \qquad \frac{d\iota}{dx} = \frac{y''}{1+y'^2}, \qquad \frac{dI}{d\iota} = \frac{dI}{dx} : \frac{d\iota}{dx} = \frac{1+y'^2}{y''} \cdot \frac{dI}{dx}.$$

In particular, taking I to be k^2 in (2), we get the differential invariant†

$$(4) \qquad \frac{d(k^2)}{d\iota} = \frac{2y'''}{(1+y'^2)^2} - \frac{6y'y''^2}{(1+y'^2)^3}.$$

In view of the equation $y = F(x)$ of the curve C, we may express the differential invariants (2) and (4) as functions of x. Assume that C is not a circle, so that the curvature is not a constant. Hence we can solve (2) for x in terms of k. Inserting the value in (4), we obtain a relation of the form

$$(5) \qquad \frac{d(k^2)}{d\iota} = \Phi(k^2).$$

Consider a second curve C_1 which is congruent to C, i. e., is derivable from C by a rigid motion in the plane. Then since k^2 and (4) are differential invariants they have the same values at corresponding points of C and C_1. Hence (5) holds also for C_1. The converse is readily proved.‡ Hence *two plane curves, neither of which is a straight line or a circle, are congruent*

* We may replace $d\tau$ by the element of arc ds along C, if the latter is not a minimal curve.

† Taking it as I, we get by (3) a differential invariant of order 4. After $n-2$ such applications of (3), we get differential invariants of orders $2, 3, \ldots, n$. If we equate to zero the nth extension of Wf and suppress the two terms involving $\partial f/\partial x$, $\partial f/\partial y$ for the reason given above, we obtain a linear partial differential equation in $y', y'', \ldots, y^{(n)}$, having therefore exactly $n-1$ independent solutions. Hence every differential invariant of order n of a plane curve is a function of

$$k^2, \qquad \frac{d(k^2)}{d\tau}, \qquad \frac{d^2(k^2)}{d\tau^2}, \qquad \ldots, \qquad \frac{d^{n-2}(k^2)}{d\tau^{n-2}}.$$

‡ Scheffers, Theorie der Kurven, ed. 2, 1910, 1921, pp. 85–86.

if and only if the same equation (5) *holds between the differential invariants of the second and third orders of both curves.*

Equation (5) is called the *intrinsic* equation of the curve since it implies all those properties of the curve which are independent of the position of the axes of coordinates.

The analogous theory for space curves is almost as simple.* But the criteria for the congruence of two surfaces are more complicated.† Lie developed‡ a theory of equivalence of n-dimensional manifolds under a continuous group.

55. **Algebraic invariants and covariants.** We shall first treat in detail the case of a quadratic form

$$\Phi = qx^2 + 2rxy + sy^2.$$

In order to give the customary definition of invariants and covariants of Φ, we apply to Φ the substitution

$$(6) \qquad x = ax_1 + by_1, \qquad y = cx_1 + dy_1, \qquad D \equiv ad - bc \neq 0,$$

and obtain the form

$$\Phi_1 = q_1x_1^2 + 2r_1x_1y_1 + s_1y_1^2,$$

in which we have employed the abbreviations

$$(7) \qquad \begin{aligned} q_1 &= qa^2 + 2rac + sc^2, \qquad r_1 = qab + r(ad + bc) + scd, \\ s_1 &= qb^2 + 2rbd + sd^2. \end{aligned}$$

By a direct computation, or by the later shorter proof,

$$r_1^2 - q_1s_1 = D^2(r^2 - qs).$$

For this reason the discriminant $r^2 - qs$ of Φ is called an *invariant* of *index* 2 of Φ.

We shall say that a function $C \equiv C(x, y, q, r, s)$ has the *covariant property* with respect to a particular substitution (6) if

$$(8) \qquad C(x_1, y_1, q_1, r_1, s_1) \equiv D^\lambda C(x, y, q, r, s),$$

* *Ibid.*, pp. 269–287.

† Scheffers, Theorie der Flächen, 1902, pp. 351–3.

‡ Lie-Scheffers, Continuierliche Gruppen, 1893, pp. 747–764.

identically in x_1, y_1, q, r, s, after q_1, r_1, s_1 have been replaced by their values (7), and x, y by their values (6). We shall call C a *covariant* of *index* λ of Φ if it has this covariant property for every substitution (6). In case C lacks x and y, it is called an *invariant* of Φ. For example, Φ itself is a covariant of Φ since (8) becomes $\Phi_1 = \Phi$.

The substitution (6) is the product of the following two:

$$x = \frac{a}{k}x_1 + \frac{b}{k}y_1, \qquad y = \frac{c}{k}x_1 + \frac{d}{k}y_1, \tag{9}$$

$$x = kx_1, \qquad y = ky_1, \tag{10}$$

where $k = \sqrt{D}$. Hence C is a covariant if and only if it has the covariant property with respect to every substitution (9) of determinant unity and also with respect to every substitution (10). The latter replaces Φ by $qk^2x_1^2 + 2rk^2x_1y_1 + sk^2y_1^2$. Hence, by (8), C has the covariant property with respect to every substitution (10) if and only if

$$C(x_1, y_1, qk^2, rk^2, sk^2) = (k^2)^\lambda C(kx_1, ky_1, q, r, s),$$

identically in x_1, y_1, q, r, s, k. This is true if and only if all terms of $C(x, y, q, r, s)$ are of the same total order ω in x and y, and of the same total degree δ in q, r, s, and then $\lambda = \frac{1}{2}(2\delta - \omega)$.

THEOREM 1. *A covariant of Φ is a function which is homogeneous in x and y and homogeneous in the coefficients of Φ and which has the covariant property with respect to all substitutions of determinant unity.*

We shall therefore first seek the functions having the covariant property with respect to every substitution (6) of determinant unity. The solved form of (6) is then

$$x_1 = dx - by, \qquad y_1 = -cx + ay, \qquad ad - bc = 1. \tag{11}$$

These equations define a transformation (§ 3). The set of all transformations (11) is readily verified to form a three-parameter group G, whose identity transformation is given by

$$a = d = 1, \qquad b = c = 0. \tag{12}$$

We obtain a one-parameter sub-group G_1 by taking a, b, c, d to be functions of a parameter t such that $ad - bc = 1$ and such that (12) hold

when $t = 0$. Write α, β, γ, δ for the values when $t = 0$ of the derivatives of a, b, c, d with respect to t. Differentiate $ad - bc = 1$, take $t = 0$, and apply (12); we get $\delta + \alpha = 0$. Hence by (11),

$$\left(\frac{dx_1}{dt}\right)_{t=0} = \delta x - \beta y, \qquad \left(\frac{dy_1}{dt}\right)_{t=0} = -\gamma x - \delta y,$$

so that the infinitesimal transformation which generates G_1 is

$$(\delta x - \beta y)\frac{\partial f}{\partial x} + (-\gamma x - \delta y)\frac{\partial f}{\partial y} \equiv -\gamma U_1 f - \beta V_1 f + \delta W_1 f,$$

where

$$(13) \qquad U_1 f \equiv x\frac{\partial f}{\partial y}, \qquad V_1 f \equiv y\frac{\partial f}{\partial x}, \qquad W_1 f \equiv x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y}.$$

The one-parameter groups generated by these infinitesimal transformations are composed of the respective transformations

$$(14) \qquad \begin{array}{lll} P_k: & x_1 = x, & y_1 = kx + y; \\ Q_k: & x_1 = x + ky, & y_1 = y; \\ R_k: & x_1 = kx, & y_1 = \dfrac{1}{k}y. \end{array}$$

According as $a \neq 0$ or $a = 0$, (11) is equal to the product

$$P_{-\frac{c}{a}}\, Q_{-fa}\, R_{\frac{1}{a}}, \qquad S Q_{\frac{d}{b}}\, R_b.$$

where $S = Q_{-1} P_1 Q_{-1}$ denotes $x_1 = -y$, $y_1 = x$. Hence a function which has the covariant property with respect to each of the infinitesimal transformations (13) will have that property with respect to group C.

By means of the solved form (6) of equations (11) we obtain from Φ the coefficients (7) of Φ_1. The transformation (11) is said to induce the transformation (7) on the coefficients q, r, s of Φ. The combined transformation (11) and (7) on the five variables x, y, q, r, s will be called a *total transformation*. Thus by (6) with $D = 1$, a function C has the covariant property with respect to a substitution (6) of determinant unity if and only if C is invariant under the corresponding total transformation (11) and (7). Here the term invariant has the sense defined in § 24 and used in the preceding chapters. Hence we seek the functions C which are invariant under each of the following three total infinitesimal transformations

$$Uf = x\frac{\partial f}{\partial y} - 2r\frac{\partial f}{\partial q} - s\frac{\partial f}{\partial r},$$

$$(15) \qquad Vf = y\frac{\partial f}{\partial x} - q\frac{\partial f}{\partial r} - 2r\frac{\partial f}{\partial s},$$

$$Wf = x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} - 2q\frac{\partial f}{\partial q} + 2s\frac{\partial f}{\partial s}.$$

To compute Uf, for example, we recall that $U_1 f$ was obtained from (11) by assuming that, when $t = 0$, the derivatives of a, b, d are zero, the derivative of c is -1, and that $a = d = 1$, $b = c = 0$. Hence, by (7),

$$\left(\frac{dq_1}{dt}\right)_{t=0} = -2r, \qquad \left(\frac{dr_1}{dt}\right)_{t=0} = -s, \qquad \left(\frac{ds_1}{dt}\right)_{t=0} = 0.$$

Thus the functions having the covariant property with respect to the group G are the solutions of

$$Uf = 0, \qquad Vf = 0, \qquad Wf = 0.$$

The latter form a complete system since

$$(16) \quad (UV)f = Wf, \qquad (UW)f = -2Uf, \qquad (VW)f = 2Vf.$$

Our complete system of three equations in five variables has exactly 5–3 or 2 independent solutions (§ 34). One solution, Φ itself, was known in advance. The discriminant $r^2 - qs$ of Φ is easily verified to be a solution. These solutions have the homogeneity properties in Theorem 1, and hence are covariants of Φ.

THEOREM 2. *All covariants of a binary quadratic form Φ are functions of Φ and its discriminant.*

This theory is readily extended to any binary form

$$(17) \qquad \Phi = a_0 x^n + \binom{n}{1} a_1 x^{n-1} y + \cdots + \binom{n}{j} a_j x^{n-j} y^j + \cdots + a_n y^n,$$

in which binomial coefficients $\binom{n}{j}$ have been prefixed to the literal coefficients a_j. We may avoid the labor of applying the substitution (6) to Φ in order

to obtain the coefficients A_i of the new form Φ_1. For example, let the infinitesimal transformation be $U_1 f$ in (13), whence

$$\frac{dx_1}{dt} = 0, \qquad \frac{dy_1}{dt} = x_1.$$

Then $d\Phi_1/dt$ is

$$\frac{dA_0}{dt}x_1^n + \binom{n}{1}\frac{dA_1}{dt}x_1^{n-1}y_1 + \cdots + \binom{n}{1}\frac{dA_{n-1}}{dt}x_1 y_1^{n-1} + \frac{dA_n}{dt}y_1^n$$

$$+ \binom{n}{1}A_1 x_1^{n-1}\cdot x_1 + \binom{n}{2}A_2 x_1^{n-2}\cdot 2y_1 x_1 + \cdots + A_n \cdot n y_1^{n-1}x_1,$$

which is identically zero since Φ_1 is equal to Φ, which is independent of t. Since the sums by colums are zero,

$$\frac{dA_0}{dt} = -nA_1, \quad \frac{dA_1}{dt} = -(n-1)A_2, \quad \ldots \quad \frac{dA_{n-1}}{dt} = -A_n, \quad \frac{dA_n}{dt} = 0.$$

Taking $t = 0$, we obtain the first of the following three formulas:

$$Uf = x\frac{\partial f}{\partial y} - uf, \quad uf = na_1\frac{\partial f}{\partial a_0} + (n-1)a_2\frac{\partial f}{\partial a_1} + \cdots + a_n\frac{\partial f}{\partial a_{n-1}};$$

$$(18)\quad Vf = y\frac{\partial f}{\partial x} - vf, \quad vf = a_0\frac{\partial f}{\partial a_1} + 2a_1\frac{\partial f}{\partial a_2} + \cdots + na_{n-1}\frac{\partial f}{\partial a_n};$$

$$Wf = x\frac{\partial f}{\partial x} - y\frac{\partial f}{\partial y} - wf, \quad wf = na_0\frac{\partial f}{\partial a_0} + \cdots + (n-2j)a_j\frac{\partial f}{\partial a_j} + \cdots.$$

We again have (16). The complete system $Uf = 0$, $Vf = 0$, $Wf = 0$ in $n+3$ variables $x, y, a_0, \ldots, a_n$ has n independent solutions (§ 34). Hence all functions having the covariant property with respect to the group (6) are functions of n independent ones. Of the latter, $n = 2$ may be chosen free of x, y (i. e., having the invariant property), when $n > 2$, since $uf = 0$, $vf = 0$, $wf = 0$ form a complete system in the variables $a_0, \ldots, a_n$ and hence have $n+1-3$ independent solutions. In fact, these three equations are independent if $n > 2$ (but dependent* if $n-2$, which accounts for the invariant of a quadratic form Φ). Consider a solution of these three equations which is either a polynomial or an infinite series, and write it in the form

* The determinant of the coefficients in the last three columns of (18) is zero identically.

$S = P_1 + P_2 + \cdots$, where $P_1, P_2, \ldots$ are homogeneous polynomials of distinct total degrees in $a_0, \ldots, a_n$. Since $uS \equiv 0$ and $uP_1, uP_2, \ldots$ have the same distinct degrees as $P_1, P_2, \ldots$, respectively, we have $uP_1 \equiv 0$, $uP_2 \equiv 0, \ldots$. Hence each P_i is a solution of the system $u = 0$, $v = 0$, $w = 0$. Hence all solutions are functions of $n-2$ solutions each of which is a homogeneous polynomial in $a_0, \ldots, a_n$ and hence is an invariant by Theorem 1.

We saw that all functions having the covariant property with respect to the group (6) are functions of n independent ones, and have now found that $n-2$ of these may be chosen as invariants when $n > 2$. Hence we need two covariants actually involving x and y. As one of these two we may take Φ itself, and as the other we may select the Hessian H of Φ, viz.,

$$H \equiv \begin{vmatrix} \frac{\partial^2 \Phi}{\partial x^2} & \frac{\partial^2 \Phi}{\partial x \partial y} \\ \frac{\partial^2 \Phi}{\partial y \partial x} & \frac{\partial^2 \Phi}{\partial y^2} \end{vmatrix}, \tag{19}$$

which is readily verified* to be a covariant of index 2 of Φ. To show that Φ and H are independent functions when $n > 2$, take the case $\Phi = x^{n-1} y$; then

$$H = -\left(\frac{\partial^2 \Phi}{\partial x \partial y}\right)^2 = -(n-1)^2 x^{2n-4}.$$

THEOREM 3. *All covariants and invariants of a binary form of degree n $(n > 2)$ are functions of the form itself, its Hessian, and $n-2$ homogeneous polynomial invariants.*

For small values of n it is easy to solve our complete system $u = 0$, $v = 0$, $w = 0$ and hence obtain the invariants. We first solve $vf = 0$, which is the condition that f shall have the invariant property with respect to all the transformations Q_k in (14). Such an f is called a *seminvariant* of Φ. The equations of Q_{-t} may be written

$$x = x_1 + t y_1, \qquad y = y_1. \tag{20}$$

This replaces Φ by $\Phi' = a_0' x_1^n + n a_1' x_1^{n-1} y_1 + \cdots$, where

$$\left\{ \begin{aligned} a_0' &= a_0, \quad a_1' = t a_0 + a_1, \quad a_2' = t^2 a_0 + 2 t a_1 + a_2, \\ a_3' &= t^3 a_0 + 3 t^2 a_1 + 3 t a_2 + a_3, \\ a_4' &= t^4 a_0 + 4 t^3 a_1 + 6 t^2 a_2 + 4 t a_3 + a_4, \ldots. \end{aligned} \right. \tag{21}$$

* Dickson, Algebraic Invariants, 1914, pp. 11–12.

the law of formation of these equations being evident. Eliminating t between the second and third equations, we get

$$a_2' - a_1'^2/a_0' = a_2 - a_1^2/a_0,$$

so that the second member is a seminvariant; it is evidently the value of a_2' when t is chosen so that $a_1' = 0$, viz., $t = -a_1/a_0$. Similarly, when this value of t is inserted in the expressions (21) for a_3' and a_4', we obtain seminvariants.* To avoid the denominators a_0, write $A_i = a_0^{i-1} a_i'$. We get

$$\begin{aligned} A_2 &= a_0 a_2 - a_1^2, \qquad A_3 = a_0^2 a_3 - 3 a_0 a_1 a_2 + 2 a_1^3, \\ A_4 &= a_0^3 a_4 - 4 a_0^2 a_1 a_3 + 6 a_0 a_1^2 a_2 - 3 a_1^4. \end{aligned} \tag{22}$$

Since the transformation (20) with $t = -a_1/a_0$ is of determinant unity and transforms Φ into Φ' having $a_1' = 0$, any seminvariant S of Φ has the property

$$S(a_0, \ldots, a_n) = S(a_0, 0, a_2', \ldots) = S\left(a_0, 0, \frac{A_2}{a_0}, \frac{A_3}{a_0^2}, \ldots, \frac{A_n}{a_0^{n-1}}\right), \tag{23}$$

and hence is a function of $a_0, A_2, \ldots, A_n$. This follows also from the fact that they are independent solutions of $rf = 0$.

A function f of $a_0, A_2, \ldots, A_n$ is a solution of $wf = 0$ if

$$\frac{\partial f}{\partial a_0} w a_0 + \frac{\partial f}{\partial A_2} w A_2 + \cdots + \frac{\partial f}{\partial A_n} w A_n = 0. \tag{24}$$

We find that

$$\begin{aligned} w a_0 &= n a_0, & w A_2 &= 2(n-2) A_2, \\ w A_3 &= 3(n-2) A_3, & w A_4 &= 4(n-2) A_4. \end{aligned} \tag{25}$$

For $n = 2$, (24) becomes $\partial f/\partial a_0 = 0$, whence f is a function of A_2. Since A_2 is a solution also of $uf = 0$, it is invariant, and every invariant is a function of A_2.

For $n = 3$, (24) becomes

$$3 a_0 \frac{\partial f}{\partial a_0} + 2 A_2 \frac{\partial f}{\partial A_2} + 3 A_3 \frac{\partial f}{\partial A_3} = 0,$$

* For a proof of this principle, see Dickson's Algebraic Invariants, p. 47.

whose solutions are evidently the functions of $r = A_2^3/a_0^2$ and $s = A_3/a_0$. We readily verify that

$$uA_2 = (A_3 + 2a_1A_2)/a_0, \qquad uA_3 = (3a_1A_3 - 6A_2^2)/a_0,$$
$$ur = 3A_2^2A_3/a_0^3, \qquad us = -6A_2^2/a_0^2.$$

Hence if a function $f(r, s)$ is a solution of $uf = 0$,

$$s\frac{\partial f}{\partial r} - 2\frac{\partial f}{\partial s} = 0, \qquad 4dr + 2sds = 0, \qquad 4r + s^2 = \text{const.}$$

Hence every invariant of a binary cubic Φ is a function of

$$(26) \qquad \frac{4A_2^3 + A_3^2}{a_0^2} = a_0^2a_3^2 - 6a_0a_1a_2a_3 + 4a_0a_2^3 + 4a_1^3a_3 - 3a_1^2a_2^2 = D,$$

which is the discriminant of Φ, being that of the reduced cubic

$$a_0x_1^3 + 3A_2a_0^{-1}x_1y_1^2 + A_3a_0^{-2}y_1^3.$$

For $n = 4$, the solutions of (24) are evidently the functions of

$$(27) \qquad r = \frac{A_2}{a_0}, \qquad s = \frac{A_3^2}{a_0^3}, \qquad t = \frac{A_4}{a_0^2}.$$

We readily verify that

$$ur = \frac{2A_3}{a_0^2}, \qquad us = \frac{2A_3}{a_0^4}(A_4 - 9A_2^2), \qquad ut = \frac{-12A_2A_3}{a_0^3}.$$

Hence a solution $f(r, s, t)$ of $uf = 0$ must satisfy

$$\frac{\partial f}{\partial r} + (t - 9r^2)\frac{\partial f}{\partial s} - 6r\frac{\partial f}{\partial t} = 0.$$

The corresponding system of ordinary differential equations is

$$dr = \frac{ds}{t - 9r^2} = \frac{dt}{-6r}.$$

An obvious solution is $3r^2+t=I$. Elimination of t gives

$$(I-12r^2)dr=ds, \qquad Ir-4r^3-s=\text{const.}$$

Inserting the expression for I, we get

$$(28)\qquad tr-r^3-s=\frac{A_2A_4-A_2^3-A_3^2}{a_0^3}=J, \qquad I=\frac{3A_2^2+A_4}{a_0^2}.$$

Hence every invariant of the quartic is a function of

$$(29)\quad I=a_0a_4-4a_1a_3+3a_2^2, \qquad J=a_0a_2a_4-a_0a_3^2+2a_1a_2a_3-a_1^2a_4-a_2^3.$$

By (23), every invariant which is a polynomial in $a_0, \ldots, a_4$ is the quotient of a polynomial P in a_0, A_2, A_3, A_4 by a power a_0^ε of a_0. The invariant was shown to be a function $f(r, s, t)$ of the expressions (27). Without altering a_0, a_2, a_4, let us change the signs of a_1 and a_3. Then A_2 and A_4 are not altered, while A_3 is changed in sign. Since r, s, t are not altered, the expression P/a_0^ε being equal to $f(r, s, t)$ must remain unaltered, whence P involves only even powers of A_3. Thus P/a_0^ε is a sum of terms $c\,a_0^\alpha\, r^\beta\, s^\gamma\, t^\delta$, where β, γ, δ are integers ≥ 0 and α is a positive or negative integer. Since this sum is equal to a function of r, s, t only, each $\alpha=0$. Hence every polynomial invariant is equal to a polynomial in r, s, t, which is evidently expressible as a polynomial in $r, t+3r^2=I, s+r^3-tr=-J$. Being a function of I and J only, the invariant is a polynomial in I and J.

THEOREM 4. *Every (polynomial) invariant of a binary quadratic or cubic form is a (polynomial) function of its discriminant. Every (polynomial) invariant of a binary quartic form is a (polynomial) function of the two invariants* (29).

While all covariants of the binary cubic f were proved above to be functions of f, its Hessian H, and its discriminant D, it does not follow that every polynomial covariant is a polynomial in f, H, D. In fact, Df^2-4H^3 is the square of a polynomial covariant G (the Jacobian or functional determinant of f and H), while G itself is not a polynomial in f, H, D. The algebraic complete system of covariants of f contains one (and only one) additional covariant G not needed in the functionally complete system f, H, D. A like result holds for the quartic f, where now $If^2H-Jf^3-4H^3$ is the square of the Jacobian G of f and H.

For the binary quintic f the algebraic complete system contains 23 covariants, four of which are invariants, the square of the one of degree 18 being a polynomial in those of degrees 4, 8, 12. Those three, together with f and its Hessian, form the functionally complete system in accord with Theorem 3.

It is customary to speak of the operators Uf and Vf as *annihilators* of covariants. In the algebraic theory, no use is made of Wf, which is the commutator of Uf and Vf. But we found above that its employment materially simplifies the computations.

There is an interesting application of continuous groups to hypercomplex numbers. An elementary account of this application has been given* by the writer.

* On the relations between linear algebras and continuous groups. Bull. Amer. Math. Soc., vol. 22 (1915), pp. 53–61.

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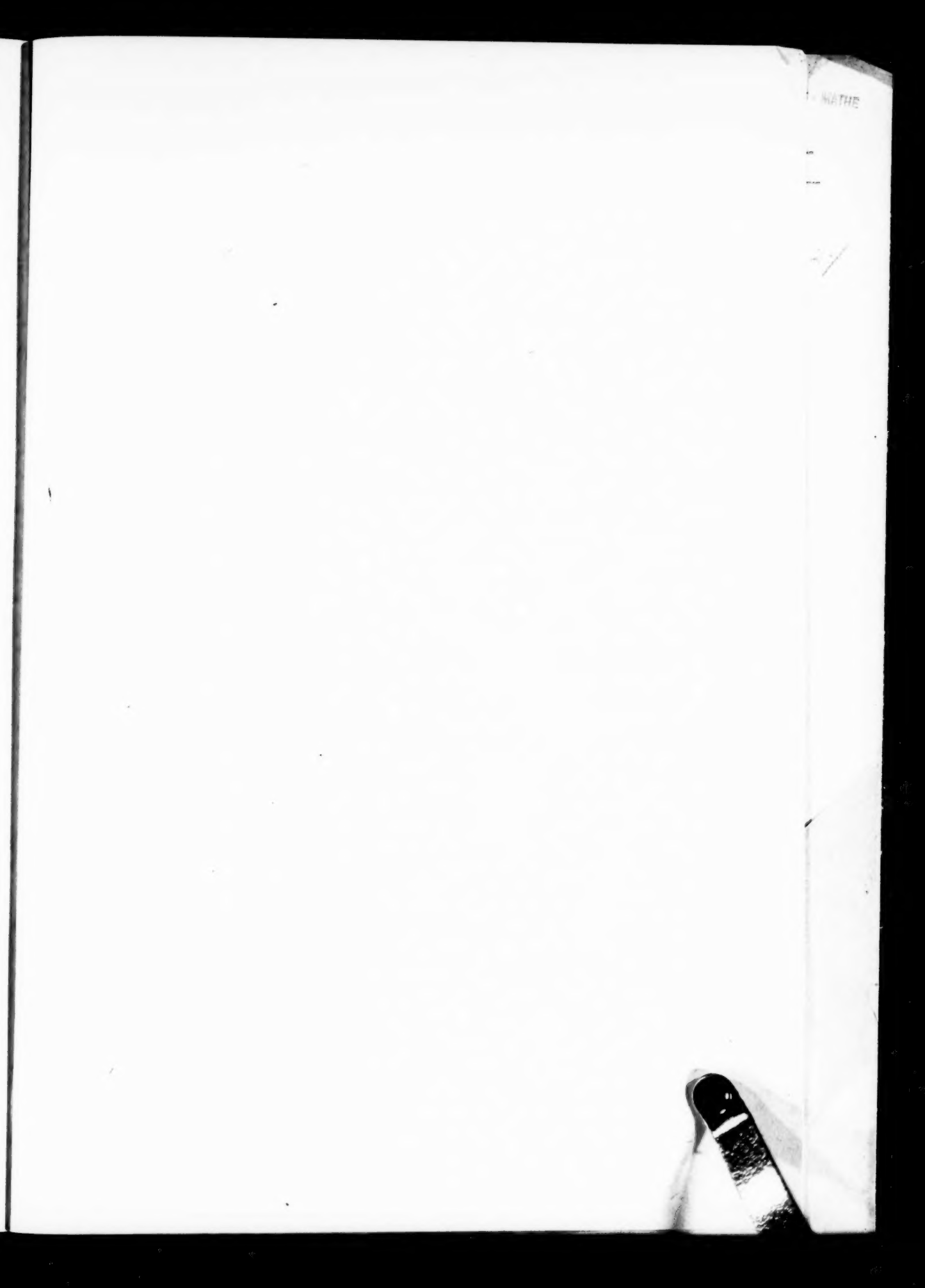
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